Advanced Engineering Mathematics

Volume II

H.C. TANEJA

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Volume 2

H.C. Taneja
Professor & Head
Dept. of Applied Mathematics
Delhi College of Engineering, Delhi

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About the Book

The complete text has been divided into two volumes: Volume I (Ch. 1-13) & Volume II (Ch. 14-22). In addition to the review material and some basic topics as discussed in the opening chapter, the main text in Volume I covers topics on infinite series, differential and integral calculus, matrices, vector calculus, ordinary differential equations, special functions and Laplace transforms. The Volume II, which is in sequel to Volume I, covers topics on complex analysis, Fourier analysis, partial differential equations and statistics.

The self-contained text has numerous distinguishing features over the already existing books on the same topic. The chapters have been planned to create interest among the readers to study and apply the mathematical tools. The subject has been presented in a very lucid and precise manner with a wide variety of examples and exercises, which would eventually help the reader for hassle free study. The book can be used as a text in Engineering Mathematics at various levels.

14 CHAPTER Functions of a Complex Variable. Analytic Functions

"The development of an analytic function is the extension of differential calculus to functions of a complex variable. The real and imaginary parts of an analytic function are solutions of Laplace equation in two dimensions and this characteristic makes them to find direct applications in two dimensional problems in elasticity, fluid mechanics and electrostatics. The conformal mapping concerns with the geometrical properties of analytic functions. Its practical utility lies in its characteristic to map region with a complicated boundary shape onto region with a simple boundary shape, facilitating in solving two dimensional boundary value problems for the Laplace equation."

14.1 SETS IN THE COMPLEX PLANE

Functions of a complex variable are defined on sets of complex number thus first we discuss various features of the sets in the complex plane.

A *point set* in the complex plane is a well-defined collection of finitely many or infinitely many points. The solutions of a quadratic equation, the points in the interior of a circle, etc. are examples of sets. Following are some most important sets which occur frequently in the study of functions of complex variables.

14.1.1 Circles, Disks and Half-plane

The equation $|z - z_0| = \delta$ defines a circle with center z_0 and radius δ. It is the set of all z whose distance from the center z_0 equals δ, refer Fig. 14.1. Any point z on this circle has the polar form $z = z_0 + \delta e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

If $z_0 = 0$ and $\delta = 1$, then the equation $|z| = 1$ defines the *unit circle* $|z| = 1$.

The *open circular disk* $|z - z_0| < \delta$ gives the interior and $|z - z_0| > \delta$ gives the exterior of the circle $|z - z_0| = \delta$.

The *closed circular disk* $|z - z_0| \leq \delta$ gives the set of all the points in the interior and the points on the circle itself.

The set of points $|z - z_0| < \delta$ is also called a *'neighbourhood'* of z_0, or in particular, an *'open circular neighbourhood'* of z_0. If we exclude the point z_0 from the open disk $|z - z_0| < \delta$, then it is called the *'deleted neighbourhood'* of the point z_0, and is written as $0 < |z - z_0| < \delta$.

The set of points $\delta_1 < |z - z_0| < \delta_2$, which lies between two concentric circles $|z - z_0| = \delta_1$ and $|z - z_0| = \delta_2$, defines an *open annulus* or *circular ring*, refer Fig. 14.2, while $\delta_1 \le |z - z_0| \le \delta_2$ defines the *closed annulus*.

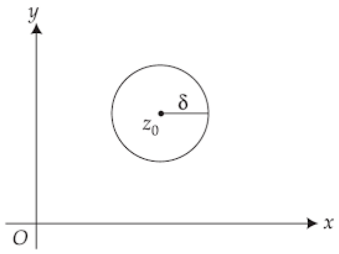


Fig. 14.1

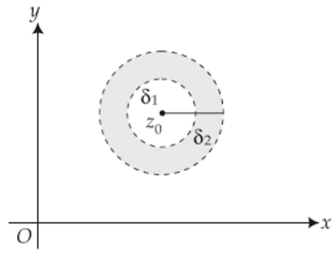


Fig. 14.2

The set of all points $z = x + iy$, $y > 0$ defines the *open upper half-plane,* refer Fig. 14.3 and the set of all points $z = x + iy$, $y < 0$ defines the *open lower half-plane*. Similarly, the condition $x > 0$ defines the *open right half-plane*, refer Fig. 14.4, and $x < 0$ defines the *open left half-plane*.

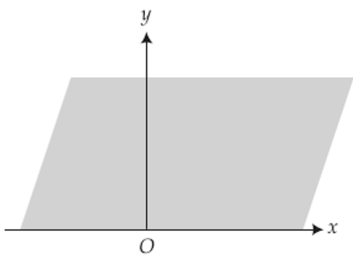


Fig. 14.3

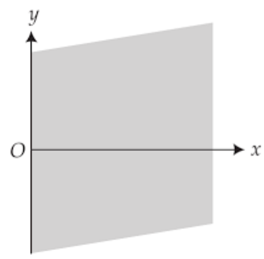


Fig. 14.4

14.1.2 Open and Closed Sets. Domain and Region

A set S is called an 'open set', if every point of S has a neighbourhood consisting entirely of points that are in S. For example, the set $S = \{z: |z - z_0| < \delta\}$, the points in the interior of a circle is an open set.

A set S is called 'closed' if its complement is open. For example, the set

$$S = \{z: |z - z_0| \ge \delta\},$$

the points in the exterior and the points on the circle $|z - z_0| = \delta$ is a closed set.

A set S is called 'connected', if any two of its points can be joined by a path consisting of finitely many straight line segments completely contained in S. For example, the set $\{z: Re(z) \ne 0\}$ is not connected but the set $\{z: Re(z) > 0\}$ is connected.

An open connected set is called a 'domain' and a 'region' is a set consisting of a domain together with all, one, or more, of its boundary points.

We note that '*a domain is necessarily a region but converse may not be true*'. For example, the open disk $|z - z_0| < \delta$ is domain as well as region, but the closed disk $|z - z_0| \leq \delta$ is a region and not a domain.

Finally, the complex plane to which the point at $z = \infty$ has been included is called the *extended complex plane* and the complex plane without the point at $z = \infty$ is called the *finite complex plane*.

14.2 COMPLEX FUNCTION, LIMIT, CONTINUITY AND DIFFERENTIABILITY

The complex analysis mainly deals with the complex functions that are differentiable in some domain. In this section we introduce first, complex function and then define the concepts of limit, continuity and differentiability in the complex. The discussion, in general, is analogous to that in case of calculus of a real variable. However, it will be of great practical importance to know the basic differences between the behaviours of real and complex valued functions.

14.2.1 Complex Function

In case of calculus of a real variable, we recall that a function f defined on a set S of reals is a rule which assigns a unique value $f(x)$ to every $x \in S$.

Analogously, in complex domain is a set of complex numbers, and we define a *function f* on D as *a rule that assigns to every $z \in D$ a complex number w, called the value of f at z, and we write $w = f(z)$*.

The set D on which f is defined is called the *domain of definition* of f and the set $D^\bullet$ of the corresponding values of $w(z)$, is called the *range* of f, refer Fig. 14.5. Nevertheless, the set D and $D^\bullet$ must be non-empty.

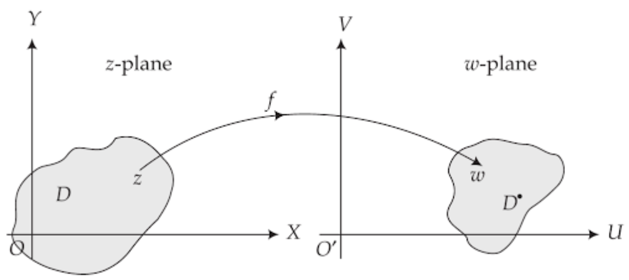


Fig. 14.5

In general, w is complex so the mapping can be depicted from the $z = (x, y)$ plane to the $w = (u, v)$ plane. Further u and v, respectively the real and imaginary parts of w, must be functions of x and y. Thus we may write

$$w = f(z) = u(x, y) + iv(x, y).$$

Hence, a complex function $f(z)$ is equivalent to a pair of real functions $u(x, y)$ and $v(x, y)$, each depending on the two real variables x and y.

If $z = r\cos\theta + ir\sin\theta = re^{i\theta}$ is taken in polar form, then the real and imaginary parts of $f(z)$ can be expressed as real valued functions of the real variables r and θ and thus we may write

$$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta).$$

If two each value of z there corresponds one and only one value of w, then the function $f(z)$ is called a *single-valued* function of z. For example, $w = 1/z$ is a single-valued function of z, defined for all values of z in the z-plane except at $z = 0$. However, in the theory of complex variable, we come across functions which take more than one value for every z belonging to the domain of definition. For example, the function $f(z) = \sqrt{z}$ is a *multi-valued* function of z, since it assumes two values for each non-zero value of z. In such cases, we restrict the discussion to those parts of the domain in which multiple-valued function behaves like a single-valued function. Each one of these single-valued functions is called a *branch* of the multiple-valued function.

Next, consider the function $f(z) = z^2$, where $z = x + iy$; $0 \le x < \infty$, $0 \le y < \infty$.

Here $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$, so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Since $0 \le x, y < \infty$, it follows from the form of u and v that $-\infty < u < \infty$ and $0 \le v < \infty$. Thus, the range of f is the entire upper half-plane $v \ge 0$, while the domain was the first quadrant. In particular, if $z = 2 + i$, then $f(z) = 3 + 4i$.

To get some insight about the graphical display of f, we consider the images of some representative curves. For example, the image of the straight line $x = 1$, $(0 \le y < \infty)$ under $f(z) = z^2$ is given parametrically by

$$u = 1 - y^2, \quad v = 2y$$

Eliminating y, we get the parabola, $v^2 = -4(u - 1)$ and similarly, the image of $y = 1$, $(0 < x < \infty)$ is given by the parabola $v^2 = 4(u + 1)$.

Similarly, we can argue that the image of the hyperbola $x^2 - y^2 = a$ is the line $v = a$ and the image of the rectangular hyperbola $2xy = b$ is the line $v = b$, where a and b are real constants.

We shall return to this aspect in detail in Section 14.6.

14.2.2 Limit and Continuity

Let $f(z)$ be a function with D as its domain of definition in the z-plane. *The function $f(z)$ is said to have limit l as z approaches a point $z_0 \in D$, if given $\varepsilon > 0$, no matter how small, we can find a $\delta(\varepsilon) > 0$, such that*

$$|f(z) - l| < \varepsilon, \text{ whenever } |z - z_0| < \delta,$$

that is, we say that $f(z)$ has the limit l as z approaches z_0, if f is defined in a neighbourhood of z_0, possibly a deleted one, and that the values of f are 'close' to l for all z 'close' to z_0, refer Fig. 14.6.

One can note that, whereas in the real case, x can approach an x_0 only along the real line thus there are only two possible paths, here in case of complex plane, z may approach z_0 from any direction. Thus in that sense the definition of limit is more strict in case of complex variable as compared to the real variable.

To define the limit of a function at $z = \infty$, we say that *the function $f(z)$ has a limit l as $z \to \infty$ if given $\epsilon > 0$, no matter how small, we can find a number $\delta(\epsilon) > 0$, such that*

$$|f(z) - l| < \epsilon, \quad \text{whenever } |z| > 1/\delta.$$

More precisely, we require that there should exist a number l with the property that given any positive number $\in$, no matter how small, there must exist a positive number δ, depending on $\in$ and possibly on z_0, such that the inequality

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - l \right| < \in$$

holds whenever z is a point in the open disk $|z - z_0| < \delta$.

We should remember that by the definition of limit $f(z)$ is defined in a neighbourhood of z_0 and z may approach z_0 from any direction in the complex plane. Hence, differentiability at z_0 means that along whatever path z approaches z_0 the $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ always approaches a certain fixed value and all these values are equal.

The various familiar rules of differentiation of functions of real variables, hold over to the complex case also. For example, we have

(a) $[f(z) + g(z)]' = f'(z) + g'(z)$

(b) $[f(z)\, g(z)]' = f'(z)\, g(z) + f(z)\, g'(z)$

(c) $\left[\dfrac{f(z)}{g(z)}\right]' = \dfrac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}, \; g(z) \neq 0$

(d) $[f(g(z))]' = f'(g(z))\, g'(z)$

Further, as in the real case, here also differentiability implies continuity but converse is not true. L'Hopital's rule also holds in case of functions of complex variables.

Example 14.1: Using the definition of limit, show that $\lim\limits_{z \to i} (z^2 + iz) = -2$

Solution: We show that given $\in > 0$, no matter how small, we can find a $\delta(\in) > 0$ such that

$$|(z^2 + iz) - (-2)| < \in, \text{ for all } z \text{ in } 0 < |z - i| < \delta$$

Consider
$$\begin{aligned} |z^2 + iz + 2| &= |(z - i)(z + 2i)| \\ &= |z - i|\ |z + 2i| \\ &= |z - i|\ |z - i + 3i| \\ &\leq |z - i|\ (|z - i| + 3) \end{aligned}$$

(Using triangular inequality)

Thus, $\quad |z^2 + iz + 2| < \in$, if $|z - i|\ (|z - i| + 3) < \in$

That is, if $\quad |z - i| < \dfrac{-3 + \sqrt{9 + 4\in}}{2}$

Choosing $\delta(\in) \leq \dfrac{-3 + \sqrt{9 + 4\in}}{2}$ and with this choice of δ, we find that $|z^2 + iz - (-2)| < \in$, whenever $0 < |z - i| < \delta$, and thus, $\lim\limits_{z \to i} (z^2 + iz) = -2$.

In fact at $z = i$, $f(z) = f(i) = -1 -1 = -2$, so the function $f(z) = z^2 + iz$ is continuous at $z = i$.

Example 14.2: Using the definition of limit show that $\lim_{z\to\infty} (1/z^2) = 0$

Solution: We show that given $\in > 0$, no matter how small, we can find a $\delta(\in) > 0$, such that $\left|\frac{1}{z^2}\right|$ $< \in$, whenever $|z| > \frac{1}{\delta}$

Now $\left|\frac{1}{z^2}\right| < \in$ implies $|z| > \frac{1}{\sqrt{\in}}$. Thus choosing $\delta < \sqrt{\in}$ and with this choice of δ we find that $\left|\frac{1}{z^2}\right| < \in$, whenever $|z| > \frac{1}{\delta}$ and thus, $\lim_{z\to\infty}\left|\frac{1}{z^2}\right| = 0$.

Example 14.3: Evaluate

(a) $\lim_{z\to i}\frac{z^2+1}{z-1}$ (b) $\lim_{z\to\infty}\left[\sqrt{z-2i}-\sqrt{z-i}\right]$

Solution: (a) $\lim_{z\to i}\frac{z^2+1}{z-1} = \lim_{z\to i}\frac{(z+i)(z-i)}{z-i} = \lim_{z\to i}(z+i) = 2i$

(b) $\lim_{z\to\infty}\left[\sqrt{z-2i}-\sqrt{z-i}\right] = \lim_{z\to\infty}\frac{\left(\sqrt{z-2i}-\sqrt{z-i}\right)\left(\sqrt{z-2i}+\sqrt{z-i}\right)}{\sqrt{z-2i}+\sqrt{z-i}}$

$$= \lim_{z\to\infty}\left[\frac{-i}{\sqrt{z-2i}+\sqrt{z-i}}\right] = \lim_{\xi\to 0}\left[\frac{-i\sqrt{\xi}}{\sqrt{1-2i\xi}+\sqrt{1-i\xi}}\right] = \frac{0}{2} = 0$$

Example 14.4: Show that the following limits do not exist

(a) $\lim_{z\to 0} f(z)$, where $f(z) = \begin{cases}\frac{\mathrm{Re}(z)}{|z|}, & z \neq 0\\ 0 & z = 0\end{cases}$ (b) $\lim_{z\to 0}\left[\frac{1}{1-e^{1/x}} + iy^2\right]$

Solution: (a) First let z move along x-axis, that is, $y = 0$ and then $x \to 0$; and second let z move along y-axis, that is, $x = 0$ and then $y \to 0$.

In the first case,

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{\mathrm{Re}(z)}{|z|} = \lim_{x\to 0}\frac{x}{|x|} = \pm 1$$

Solution: For any point z in the complex plane,

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Hence $f(z)$ is differentiable everywhere in the finite complex plane.

In fact we can prove that z^n, where n is a positive integer, is differentiable everywhere in the finite complex plane and hence any polynomial function $f(z)$ of the form

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n,$$

where a_i's are complex constants is continuous everywhere in the complex plane.

Example 14.7: Show that the function $f(z) = \overline{z}$ does not have a derivative at any point.

Solution: We observe that

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Consider the path $y = mx$. We have $\Delta y = m\Delta x$. Here $\Delta x \to 0$ implies $\Delta y \to 0$ and thus $\Delta z \to 0$. Consider

$$\lim_{z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x} = \frac{1 - im}{1 + im}$$

which depends on m. Thus the limit does not exist and, therefore, the function $f(z) = \overline{z}$ is not differentiable anywhere.

Example 14.8: Show that the function $f(z) = |z|^2$ is differentiable only at $z = 0$.

Solution: We have $f(z) = |z|^2 = z\,\overline{z}$, therefore,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z + \Delta z}) - z\overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z\overline{\Delta z} + \overline{z}\Delta z + \Delta z\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[z\frac{\overline{\Delta z}}{\Delta z} + \overline{z} + \overline{\Delta z} \right] = 0 \text{ at } z = 0.$$

Hence the function $f(z) = |z|^2$ is differentiable at $z = 0$ and $f'(0) = 0$.

For $z \neq 0$, $\lim_{\Delta z \to 0} \left[\overline{z} + \overline{\Delta z}\right] = \overline{z}$ but, $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist, refer Example 14.7.

Hence, the function $f(z) = |z|^2$ is not differentiable at $z \neq 0$.

Example 14.9: If $f(z) = \begin{cases} x^3 y(y - ix)/(x^6 + y^2) & z \neq 0 \\ 0 & z = 0 \end{cases}$

prove that $[f(z) - f(0)]/z \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ along the curve $y = mx^3$.

6. Find the following limits, if it exists

(a) $\lim_{z \to 1-i} [z^2 - \bar{z}^2]$ (b) $\lim_{z \to \infty} \sqrt{z}\left[\sqrt{z-2i} - \sqrt{z-i}\right]$

(c) $\lim_{z \to 0} \frac{z^2}{|z|^2}$ (d) $\lim_{z \to 0} \frac{\text{Re}(z).\text{Im}(y)}{|z|^2}$

7. If $f(x)$ is continuous for all real x, does it follow that $f(z)$ is continuous everywhere in the z-plane? Explain.

8. Check the continuity of the following functions at the origin

(a) $f(z) = \frac{xy^3}{x^2 + y^6}$ $(z \neq 0)$, $f(0) = 0$ (b) $f(z) = \frac{xy}{x^2 + y^2}$ $(z \neq 0)$, $f(0) = 0$

(c) $f(z) = \frac{x^2}{(x^2 + y^2)^{1/2}}$ $(z \neq 0)$, $f(0) = 0$

9. Show that the function $f(z) = \bar{z}$ is continuous at the point $z = 0$ but not differentiable at $z = 0$.

10. Using the definition of derivative obtain $f'(z)$, if it exists, for the following functions.

(a) $f(z) = \frac{1}{z+1}$ $(z \neq -1)$ (b) $f(z) = \frac{1}{z^2}$, $z \neq 0$.

(c) $f(z) = \bar{z}^2$ (d) $f(z) = \frac{1+z}{1-z}$, $z \neq 1$

11. Prove that if $f(z)$ is differentiable at z_0, then it must be continuous there.

14.3 BASIC ELEMENTARY COMPLEX FUNCTIONS

In this section we discuss a few basic elementary complex functions like the exponential functions, trigonometric functions, hyperbolic functions, logarithmic function, inverse trigonometric and inverse hyperbolic functions. They reduce to their counterparts in real calculus when $z = x$ is real and some of these have interesting properties and applications.

14.3.1 Exponential Function

We define the exponential function of the complex variable z as an extension of the corresponding function of the real variable x. We know that the Maclaurin series for the exponential function e^x for real x is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \ldots(14.1)$$

Thus we define the exponential function e^z for complex z as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \qquad \ldots(14.2)$$

Since (14.2) reduces to (14.1) when z is purely real, we formally write

$$\begin{aligned} e^z &= e^{x+iy} = e^x e^{iy} \\ &= e^x\left[1 + (iy) + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \ldots\right] \quad \text{(using 14.2)} \\ &= e^x\left\{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)\right\}. \end{aligned}$$

Using the Maclaurin series expansion of sin y and cos y, this gives

$$e^z = e^x (\cos y + i \sin y) \qquad \ldots(14.3)$$

The formula

$$e^{iy} = (\cos y + i \sin y) \qquad \ldots(14.4)$$

is known as *Euler's formula* and is a special case of (14.3) when z is purely imaginary and is very useful.

Since $|e^{iy}| = |\cos y + i \sin y| = 1$, thus

$$|e^z| = |e^{x+iy}| = e^x |e^{iy}| = e^x \neq 0,$$

for all finite x. This implies that e^z is non-zero for all finite z. Also the

$$\arg e^z = y \pm 2n\pi, \quad n = 0, 1, 2, \ldots$$

Further as in real, we have

$$e^{z_1 + z_2} = e^{z_1} e^{z_2},$$

since for any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1} [\cos y_1 + i \sin y_1]\ e^{x_2} [\cos y_2 + i \sin y_2] \\ &= e^{x_1 + x_2} [(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \sin y_2 + \cos y_2 \sin y_2)] \\ &= e^{x_1 + x_2} [\cos (y_1 + y_2) + i \sin (y_1 + y_2)] = e^{z_1 + z_2}. \end{aligned}$$

The domain of definition of e^z is the whole of the complex plane and the range is also the whole complex plane except the origin.

The function is periodic with period $2\pi i$, since $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$ and thus we can write $e^{z + 2n\pi i} = e^z$. Also e^z, $z = x + iy$, is real for any x and $y = n\pi$ and e^z is pure imaginary for any x and $y = (2n + 1)\pi/2$, where n is any integer.

The polar form of a complex number $z = r[\cos \theta + i \sin \theta]$ in terms of exponential function is written as $z = re^{i\theta}$.

14.3.2 Trigonometric Functions

The Euler's formula (14.4) is

$$e^{iy} = \cos y + i \sin y$$

Changing y to $-y$, we obtain

$$e^{-iy} = \cos y - i \sin y$$

Solving for $\cos y$ and $\sin y$, we obtain

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \qquad ...(14.5)$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \qquad ...(14.6)$$

Analogous to expressions (14.5) and (14.6), we define the cosine and sine functions of a complex variable z as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad ...(14.7)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad ...(14.8)$$

Besides $\sin z$ and $\cos z$, the other trigonometric functions are defined as

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z},$$

whenever the denominator is not zero.

Further (14.7) and (14.8) imply that Euler's formula (14.4) is valid in complex also, that is,

$$e^{iz} = \cos z + i \sin z \qquad ...(14.9)$$

Next, we investigate for the real and imaginary parts of $\cos z$ and $\sin z$. These are helpful in investigating the properties of these functions further.

We have

$$\cos z = \frac{1}{2}[e^{iz} + e^{-iz}] = \frac{1}{2}[e^{i(x+iy)} + e^{-i(x+iy)}]$$

$$= \frac{1}{2}[e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)]$$

$$= \left[\cos x\left(\frac{e^{y} + e^{-y}}{2}\right) - i \sin x\left(\frac{e^{y} - e^{-y}}{2}\right)\right]$$

Thus,

$$\cos z = \cos x \cosh y - i \sin x \sinh y \qquad ...(14.10)$$

Similarly, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y \qquad \text{...(14.11)}$$

From (14.10)

$$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$
$$= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y$$
$$= \cos^2 x + (\cos^2 x + \sin^2 x) \sinh^2 y$$

Thus, $$|\cos z|^2 = \cos^2 x + \sinh^2 y \qquad \text{...(14.12)}$$

Similarly, we can show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \qquad \text{...(14.13)}$$

Formulae (14.12) and (14.13) point to an essential difference between the real and the complex cosine and sine. Since $\sinh y = \dfrac{e^y - e^{-y}}{2}$ tends to ∞ as y tends to ∞, thus $|\sin z|$, $|\cos z|$ and hence $\sin z$ and $\cos z$ are not bounded whereas their counterparts $\sin x$ and $\cos x$ are bounded.

Also we observe from (14.7) and (14.8) that $\cos z$ and $\sin z$ are periodic with period 2π just as in real and $\tan z$ is periodic with period π.

Further, $\sin z = 0$ implies that $|\sin z| = 0$, which gives

$$\sin^2 x + \sinh^2 y = 0, \text{ for all real } x, y.$$

or, $$\sin x = 0 \text{ and } \sinh^2 y = 0$$

or, $$x = n\pi \text{ and } y = 0, \text{ that is, } z = n\pi.$$

Hence, $\sin z = 0$ only when z is purely real and $z = n\pi$, n any integer.

Similarly, $\cos z = 0$ only when z is real and $z = (2n + 1)\pi/2$, n any integer.

The general formulae for the real trigonometric functions continue to hold for the corresponding complex valued functions also and can be easily verified using (14.7) and (14.8). We mention the following

(a) $\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

(b) $\cos (z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

(c) $\tan (z_1 \pm z_2) = (\tan z_1 + \tan z_2)/(1 \mp \tan z_1 \tan z_2)$

(d) $\sin 2z = 2 \sin z \cos z$

(e) $\cos 2z = \cos^2 z - \sin^2 z$

(f) $\sin^2 z + \cos^2 z = 1$

14.3.3 Hyperbolic Functions

The complex hyperbolic cosine and sine functions are defined as

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2} \qquad \text{...(14.14)}$$

The other hyperbolic functions are defined by

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \text{sech } z = \frac{1}{\cosh z}, \quad \text{cosech } z = \frac{1}{\sinh z}$$

whenever the denominator is not zero.

If we replace z by iz in (14.14), then we have

$$\cosh iz = \cos z \quad \text{and} \quad \sinh iz = i \sin z \qquad \text{...(14.15a)}$$

Again replacing z by iz in (14.15a) and using the fact that $\cosh z$ is an even function of z and $\sinh z$ is an odd function of z, we obtain

$$\cos iz = \cosh z \quad \text{and} \quad \sin iz = i \sinh z \qquad \text{...(14.15b)}$$

Also, $$\sinh z = \frac{1}{i} \sin iz = -i \sin iz = -i \sin i(x + iy)$$

$$= i \sin (y - ix) = i[\sin y \cosh x - i \cos y \sinh x]$$

or, $$\sinh z = \sinh x \cos y + i \cosh x \sin y \qquad \text{...(14.16)}$$

Similarly,

$$\cosh z = \cosh x \cos y + i \sinh x \sin y \qquad \text{...(14.17)}$$

Also from (14.16), we have

$$\begin{aligned} |\sinh z| &= \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y \\ &= \sinh^2 x(1 - \sin^2 y) + \cosh^2 x \sin^2 y \\ &= \sinh^2 x + (\cosh^2 x - \sinh^2 x) \sin^2 y \\ &= \sinh^2 x + \sin^2 y, \text{ using } \cosh^2 x - \sinh^2 x = 1. \end{aligned}$$

Similarly from (14.17), we have

$$|\cosh z| = \sinh^2 x + \cos^2 y$$

Next, $\sinh z = 0$ gives $|\sinh z| = 0$, or $\sinh^2 x = 0$ and $\sin y = 0$, or $x = 0$ and $y = n\pi$.

Thus $\sinh z = 0$, only when $z = n\pi i$ is purely imaginary. Similarly, $\cosh z = 0$, only when $z = (2n + 1)\frac{\pi}{2} i$, where n is any integer.

Further, as their counterparts $\sinh x$ and $\cosh x$, $\sinh z$ and $\cosh z$ are also not bounded. Also the general formulae for the real hyperbolic functions continue to hold for the corresponding complex valued functions also and can be easily verified. We mention the following

(a) $\sinh (z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(b) $\cosh (z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(c) $\tanh (z_1 \pm z_2) = \dfrac{\tanh z_1 + \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

(d) $\cosh^2 z - \sinh^2 z = 1$

(e) $\sinh 2z = 2\sinh z \cosh z$

(f) $\cosh 2z = \cosh^2 z + \sin h^2 z$

14.3.4 Logarithmic Function

The natural logarithm of $z = x + iy$ denoted by $\ln z$, or sometime by $\log_e z$, is defined as the inverse of the exponential function, that is, if $z = e^w$, then $w = \ln z$, $z \neq 0$, since $e^w \neq 0$ for all w.

Writing $z = re^{i\theta}$ and $w = u + iv$, we have

$$re^{i\theta} = e^{u + iv}$$

Since, $e^{i\theta} = e^{i(\theta + 2n\pi)}$ for any integer n, thus it can be written as

$$re^{i(\theta + 2n\pi)} = e^{u + iv} \qquad \text{...(14.18)}$$

Now $e^{u + iv}$ has absolute value e^u and the argument v, then comparing these on both sides of (14.18) we have $e^u = r$, or $u = \ln r$ and $v = \theta + 2n\pi$, n any integer.

Hence for any complex $z \neq 0$, the solutions of equation $e^w = z$ are given by

$$w = \ln z = \ln r + i(\theta + 2n\pi), \quad n \text{ any integer}$$

$$= \ln r + i \arg(z), z \neq 0$$

Thus the complex natural logarithm $\ln z$, $(z \neq 0)$ is infinitely multiple-valued function. For each n, we obtain a different branch of the multiple-valued function $\ln z$. If we restrict $\arg(z)$ to its principal value, $-\pi < \arg(z) \leq \pi$, denoted by Arg (z), then the corresponding branch Ln z of $\ln z$ is

$$Ln\, z = \ln r + i \operatorname{Arg}(z)$$

or,
$$Ln\, z = \ln |z| + i\theta, -\pi < \theta \leq \pi$$

or,
$$Ln\, z = \ln \sqrt{x^2 + y^2} + i \tan^{-1}(y/x)$$

The uniqueness of Arg (z) for given $z \neq 0$ implies that $Ln\, z$ is single-valued function and since different values of $\arg(z)$ differ by integer multiple of 2π, we have

$$\ln z = Ln\, z \pm 2n\pi i, \quad n = 1, 2, \ldots$$

They all have the same real part but their imaginary parts differ by an integral multiple of 2π.

If $z = x$ is positive real, then $|z| = x$ and Arg $(z) = 0$, and thus

$$Ln\, z = \ln|z| = \ln x, x > 0$$

which is the real natural logarithm.

If $z = x$ is negative real, then $|z| = |x|$ and Arg $(z) = \pi$ and thus

$$Ln(z) = \ln|z| + i\pi = \ln|x| + i\pi, x < 0$$

Also the following properties can easily be verified for natural logarithm of complex values

(a) $e^{\ln z} = z$

(b) $\ln(z_1 z_2) = \ln z_1 + \ln z_2 \pm 2n\pi i$

(c) $\ln(z_1/z_2) = \ln z_1 - \ln z_2 \pm 2n\pi i$, where $n = 0, 1, 2, \ldots$.

14.3.5 General Powers

The general power of a complex number $z = x + iy$ is defined by the formula

$$z^c = e^{c \ln z}, \; z \neq 0$$

where c is a complex constant.

Since $\ln z$ is infinitely many valued function, thus z^c will also be, in general, multiple valued function. The particular value $e^{c\, Lnz}$ is called the *principal value* of z^c.

If $c = n$ is a positive integer, then z^n is single valued and similarly for $z = -n$. But if $c = 1/n$, then z^c has exactly n values. If $c = p/q$, the quotient of two positive integers, then z^c has exactly q distinct values. If c is an irrational or a complex, then z^c is multivalued function.

14.3.6 Inverse Trigonometric Functions of a Complex Variable

If $z = \sin w$, then w is called the inverse sine function for a complex variable z and is written as $w = \sin^{-1} z$.

The function $\sin w = z$ implies $e^{iw} - e^{-iw} = 2iz$ or, $e^{2iw} - 2ize^{iw} - 1 = 0$

Solving for e^{iw} we get, $e^{iw} = iz \pm \sqrt{1-z^2}$. Since the double sign is covered by the double-valued function $\sqrt{1-z^2}$, thus

$$e^{iw} = iz + \sqrt{1-z^2} \quad \text{or, } w = -i \ln\left[iz + \sqrt{1-z^2}\right]$$

Hence, $$\sin^{-1} z = -i \ln\left[iz + \sqrt{1-z^2}\right]$$

Now, $\sin^{-1} z$ is defined for all z except when

$$iz = -\sqrt{1-z^2} \quad \text{or, } -z^2 = 1 - z^2 \quad \text{or, } 1 = 0,$$

which is not possible. Thus $\sin^{-1} z$ is defined for all z.

The other complex inverse trigonometric functions are given by

$$\cos^{-1} z = -i \ln\left(z + \sqrt{z^2-1}\right)$$

$$\tan^{-1} z = \frac{i}{2} \ln \frac{i+z}{i-z}, \; z \neq \pm i$$

$$\operatorname{cosec}^{-1} z = \sin^{-1}\left(\frac{1}{z}\right) = -i \ln\left[\frac{1+\sqrt{z^2-1}}{z}\right], \; z \neq 0$$

$$\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right) = -i \ln\left[\frac{1+\sqrt{1-z^2}}{z}\right], \; z \neq 0$$

$$\cot^{-1} z = \tan^{-1}\left(\frac{1}{z}\right) = -\frac{i}{2}\ln\left[\frac{z+i}{z-i}\right], z \neq \pm i$$

Clearly inverse trigonometric functions are multivalued functions but we shall consider only their principal values.

14.3.7 Inverse Hyperbolic Functions of a Complex Variable

If $z = \sinh w$, then w is called the inverse sine hyperbolic function of the complex variable z and is written as $w = \sinh^{-1} z$.

The function $\sinh w = z$ implies $e^{w} - e^{-w} = 2z$, or $e^{2w} - 2ze^{w} - 1 = 0$

Solving for e^{w} we get $e^{w} = z \pm \sqrt{z^2+1}$. Since the double sign is covered by the double valued function $\sqrt{z^2+1}$, thus

$$e^{w} = z + \sqrt{z^2+1} \text{ or, } w = \ln\left[z + \sqrt{z^2+1}\right]$$

Hence, $$\sinh^{-1} z = \ln\left[z + \sqrt{z^2+1}\right]$$

The other inverse hyperbolic trigonometric functions of the complex variable are given by

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \quad z \neq \pm 1$$

$$\operatorname{cosech}^{-1} z = \sinh^{-1}\left(\frac{1}{z}\right) = \ln\left\{\frac{1+\sqrt{1+z^2}}{z}\right\}, \quad z \neq 0$$

$$\operatorname{sech}^{-1} z = \cosh^{-1}\left(\frac{1}{z}\right) = \ln\left\{\frac{1+\sqrt{1-z^2}}{z}\right\}, \quad z \neq 0$$

$$\coth^{-1} z = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right), \quad z \neq \pm 1.$$

Clearly inverse hyperbolic trigonometric functions are also multivalued functions but we shall consider only their principal values.

Example 14.10: Solve $e^{z} = 4 + 3i$.

Solution: We have, $e^{z} = e^{x+iy} = e^{x}(\cos y + i \sin y) = 4 + 3i$

Therefore, $e^{x} \cos y = 4$ and $e^{x} \sin y = 3$. Squaring and adding, we have

$$e^{2x} = 25 \text{ or, } e^{x} = 5 \text{ or, } x = 1.609$$

Also, $\tan y = \frac{3}{4}$, or $y = \tan^{-1}\frac{3}{4}$, or $y = 0.6435$

Hence, $z = 1.609 + 0.6435\,i \pm 2n\pi i$, where $n = 0, 1, 2, \ldots$

Example 14.11: Solve $\cos z = 5$

Solution: The equation $\cos z = 5$ gives

$$\frac{e^{iz} + e^{-iz}}{2} = 5, \text{ or } e^{2iz} - 10e^{iz} + 1 = 0$$

Solving for e^{iz} we get $e^{iz} = 5 \pm \sqrt{25-1} = 9.899$, or 0.101

Now, $e^{iz} = 9.899$ gives $e^{-y+ix} = 9.899$, or $e^{-y}[\cos x + i \sin x] = 9.899$

This implies $e^{-y} = 9.899$, or $y = -\ln 9.899 = -2.292$. Also $\cos x = 1$ and $\sin x = 0$, this gives $x = \pm 2n\pi$.

Similarly, $e^{iz} = 0.101$ gives $e^{-y+ix} = 0.101$, which implies

$y = -\ln(0.101) = 2.292$ and $x = \pm 2n\pi$, and hence

$z = \pm 2n\pi \pm 2.292i$, where n is any integer.

Example 14.12: Solve $\cosh z = \frac{1}{2}$

Solution: The equation $\cosh z = \frac{1}{2}$ gives

$$\frac{e^{z} + e^{-z}}{2} = \frac{1}{2}, \text{ or } e^{2z} - e^{z} + 1 = 0$$

Solving for e^z, we get $e^z = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$,

or, $e^x[\cos y + i \sin y] = (1 \pm i\sqrt{3})/2$. Comparing we get

$$e^x \cos y = 1/2 \text{ and } e^x \sin y = \pm\sqrt{3}/2,$$

which gives $e^x = 1$, that is, $x = 0$

and, $\cos y = 1/2$ and $\sin y = \pm\sqrt{3}/2$, which give $y = \pm\pi/3$

Hence solutions are $z = i(\pm\pi/3 \pm 2n\pi)$, where n is any integer.

Example 14.13: Solve $\tan z = e^{i\alpha}$, where α is a real.

Solution: Let $z = x + iy$. The equation $\tan z = e^{i\alpha}$ gives

$\tan(x + iy) = \cos\alpha + i \sin\alpha,$

$\tan(x - iy) = \cos\alpha - i \sin\alpha$

Therefore, $\tan 2x = \tan[(x+iy)+(x-iy)] = \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$$= \frac{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)}{1-(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1-(\cos^2\alpha+\sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

Thus, $2x = \dfrac{\pi}{2} + n\pi$ or $x = \left(n+\dfrac{1}{2}\right)\dfrac{\pi}{2}$, n being any integer.

Also, $\tan 2iy = \tan[(x+iy)-(x-iy)] = \dfrac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$

$$= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1+(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

or, $i\tanh 2y = \dfrac{2i\sin\alpha}{1+(\cos^2\alpha+\sin^2\alpha)} = i\sin\alpha$

or, $\dfrac{e^{2y}-e^{-2y}}{e^{2y}+e^{-2y}} = \dfrac{\sin\alpha}{1}$

By componendo and dividendo, we get

$$\frac{e^{2y}}{e^{-2y}} = \frac{1+\sin\alpha}{1-\sin\alpha} = \frac{\cos^2\alpha/2+\sin^2\alpha/2+2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2+\sin^2\alpha/2-2\sin\alpha/2\cos\alpha/2}$$

or, $e^{4y} = \dfrac{(\cos\alpha/2+\sin\alpha/2)^2}{(\cos\alpha/2-\sin\alpha/2)^2}$

or, $e^{2y} = \dfrac{1+\tan\alpha/2}{1-\tan\alpha/2} = \tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)$ or, $y = \dfrac{1}{2}\ln\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)$

Hence solution is

$$z = \left(n+\frac{1}{2}\right)\frac{\pi}{2} + \frac{i}{2}\ln\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Example 14.14: Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.

Solution: Let $\cos^{-1}(\cos\theta + i\sin\theta) = u + iv$, then

$$\cos\theta + i\sin\theta = \cos(u + iv) = \cos u\cosh v - i\sin u\sinh v$$

Therefore,

$$\cos\theta = \cos u\cosh v \qquad ...(14.19)$$

$$\sin\theta = -\sin u\sinh v \qquad ...(14.20)$$

Squaring and adding, we have

$$1 = \cos^2 u\cosh^2 v + \sin^2 u\sinh^2 v = \cos^2 u\,(1 + \sinh^2 v) + \sin^2 u\sinh^2 v$$

$$= \cos^2 u + (\cos^2 u + \sin^2 u)\sinh^2 v = \cos^2 u + \sinh^2 v$$

or,

$$1 - \cos^2 u = \sinh^2 v, \text{ or } \sin^2 u = \sinh^2 v \qquad ...(14.21)$$

Squaring (14.20) and using (14.21) we obtain $\sin\theta = \sin^2 u$, θ being a positive acute angle $\sin\theta$ is positive. Thus, the real part is

$$u = \sin^{-1}\sqrt{\sin\theta}$$

Using $\sin u = \sqrt{\sin\theta}$ in (14.20), we have, $\sinh v = -\sqrt{\sin\theta}$; the imaginary part is

$$v = \sinh^{-1}\left(-\sqrt{\sin\theta}\right) = \ln\left[-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right] = \ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$$

Hence, $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$.

Example 14.15: Compute i^i.

Solution: By definition $i^i = e^{i\ln i}$

But, $\ln i = Ln1 + i(\pi/2 + 2n\pi) = i(\pi/2 + 2n\pi)$, and hence

$i^i = e^{-(\pi/2 + 2n\pi)}$, where n is any integer.

Example, 14.16: Find all the roots of the equation

(a) $\sin z = \cosh 4$ (b) $\sinh z = i$

Solution: (a) $\sin z = \cosh 4 = \cos 4i = \sin\left(\frac{\pi}{2} - 4i\right)$

Therefore, $z = n\pi + (-1)^n\left(\frac{\pi}{2} - 4i\right)$, since $\sin\theta = \sin\alpha$ implies $\theta = n\pi + (-1)^n\alpha$

(b) The equation is, $i = \sinh z = \frac{e^z - e^{-z}}{2}$. It gives

$$e^{2z} - 2ie^z - 1 = 0 \quad \text{or,} \quad (e^z - i)^2 = 0 \text{ or, } e^z = i$$

or,

$$z = \ln i = Ln1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(2n + \frac{1}{2}\right)\pi$$

EXERCISE 14.2

1. Verify the following:

   (a) $e^{z_1} e^{z_2} = e^{z_1 + z_2}$ (b) $\sin^2 z + \cos^2 z = 1$

   (c) $\cos (z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$ (d) $\cos iz = \cosh z$

   (e) $\sin iz = i \sinh z$

2. Compute

   (a) $\cos (2 + i)$ (b) 1^i (c) $(1 + i)^i$ (d) i^{-i}

3. Find all the values of z which satisfy the following

   (a) $\cos z = i$ (b) $\tan z = 2$

   (c) $\tanh z = -2$ (d) $\cosh z + \sinh z = \alpha$, α is complex constant.

4. Find all solutions of the following equations:

   (a) $e^z + 1 = 0$ (b) $\sin z - 2 = 0$ (c) $\cos z - 1 = 0$

5. If $\cot (\theta + i\phi) = e^{i\alpha}$, then show that

$$\theta = (2n + 1)\frac{\pi}{4} \text{ and } \phi = \frac{1}{2}\ln\left[\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right], \text{ where } n \text{ is an integer.}$$

6. If $\sin (\alpha + i\beta) = x + iy$, prove that

$$(x^2/\cosh^2 \beta) + (y^2/\sinh^2 \beta) = 1, \quad (x^2/\sin^2 \alpha) - (y^2/\cos^2 \alpha) = 1$$

7. Separate $\ln \cos (x + iy)$ into real and imaginary parts.

8. If $i^{\alpha + i\beta} = \alpha + i\beta$, then prove that $\alpha^2 + \beta^2 = e^{-(4n + 1)\pi\beta}$.

9. If $(a_1 + ib_1)(a_2 + ib_2) \ldots (a_n + ib_n) = A + iB$, prove that

   (a) $\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right) \ldots \left(a_n^2 + b_n^2\right) = A^2 + B^2$

   (b) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$

10. If $\sin^{-1}(u + iv) = \alpha + i\beta$, prove that $\sin^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation $x^2 - (1 + u^2 + v^2)x + u^2 = 0$.

14.4 ANALYTIC FUNCTIONS. CAUCHY-RIEMANN EQUATIONS

In complex analysis we are interested in the functions which are differentiable in some domain, called the analytic functions. A large variety of functions of complex variables which are useful for applications purpose are analytic. In this section we introduce analytic functions and discuss the necessary and sufficient conditions for the analyticity of a function.

14.4.1 Analytic Function

A function $f(z)$ is said to be 'analytic' at a point z_0, if it is differentiable at z_0 and, in addition, it is differentiable throughout some neighbourhood of z_0.

Thus the analyticity at a point z_0 means that the function is analytic in a neighbourhood of z_0. Further a function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. In fact, *analyticity is a 'global' property while differentiability is a 'local' property.*

The terms *regular* and *holomorphic* are also used in place of analytic.

We note that the polynomial functions

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n$$

are differentiable everywhere, and hence analytic everywhere in the complex plane. However, the function $f(z) = |z|^2$ is differentiable only at the point $z = 0$, refer Example [14.8], hence this function is analytic nowhere.

A function $f(z)$ is said to be analytic at $z = \infty$, if the function $f(1/z)$ is analytic at $z = 0$.

If a function $f(z)$ ceases to be analytic at a point $z = z_0$, then z_0 is called a *'singular point'* of the function $f(z)$. For example $z = 0$ is a singular point of the function $f(z) = \dfrac{1}{z}$.

Further, if functions $f(z)$ and $g(z)$ are analytic in D, then the functions $f(z) \pm g(z)$, $f(z)g(z)$ are also analytic in D. The rational function $f(z)/g(z)$ is also analytic in D, except at the points where $g(z) = 0$; here we assume that the common factors of f and g have been cancelled. Also, the composition of two analytic functions is also analytic.

Next we discuss the necessary and sufficient conditions for a function to be analytic.

14.4.2 Cauchy-Riemann Equations

Cauchy-Riemann equations provide a criterion for the analyticity of a complex function $f(z) = u(x, y) + iv(x, y)$.

Theorem 14.1: ***(Necessary conditions for a function to be analytic)*** *If $f(z) = u(x, y) + iv(x, y)$ is continuous in some neighbourhood of a point $z = x + iy$ and is differentiable at z, then the first order partial derivatives of $u(x, y)$ and $v(x, y)$ exist and satisfy the Cauchy-Riemann equations*

$$u_x = v_y \text{ and } u_y = -v_x \qquad \ldots(14.22)$$

at the point $z = x + iy$.

That is, if $f(z)$ is analytic in a domain D, then partial derivatives exist and satisfy the C.R. = ns at all points of D.

Proof. Since $f(z)$ is differentiable at z, we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y} \qquad \ldots(14.23)$$

and the limit on the right of (14.23) must be independent of the path along which $\Delta z \to 0$.

First set $\Delta y = 0$ in $\Delta z = \Delta x + i\Delta y$ so that $\Delta z = \Delta x$, that is, Δz tends to zero parallel to x-axis. Thus the limit on the right side of (14.23) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \ldots(14.24)$$

Next, set $\Delta x = 0$ in $\Delta z = \Delta x + i\Delta y$ so that $\Delta z = i\Delta y$, that is, $\Delta z \to 0$ parallel to the y-axis. Then the limit on the right side of (14.23) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \text{...(14.25)}$$

Since $f(z)$ is differentiable the value of the limits attained along two different paths, refer Fig. 14.7, in (14.24) and (14.25) must be equal. Therefore

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Comparing the real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point $z = (x, y)$.

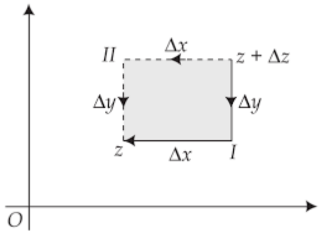


Fig. 14.7

These are known as the *'Cauchy-Riemann equations'*. Satisfaction of these equations is necessary for differentiability and analyticity of the function $f(z)$ at a given point. Thus, if a function $f(z)$ does not satisfy the Cauchy-Reimann equations at a point, it is not differentiable and hence not analytic at that point. But these conditions are not sufficient since these have been obtained by considering only two possible paths of approach to the point z. For example, consider the function

$$f(z) = \begin{cases} \dfrac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases},$$

Here, $u = \dfrac{x^3 - y^3}{x^2 + y^2}$ and $v = \dfrac{x^3 + y^3}{x^2 + y^2}$. By definition,

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{u(x, 0)}{x} = \frac{x^3}{x^3} = 1$$

Similarly, $\left.\dfrac{\partial u}{\partial y}\right|_{(0,0)} = -1, \quad \left.\dfrac{\partial v}{\partial x}\right|_{(0,0)} = 1$, and $\left.\dfrac{\partial v}{\partial y}\right|_{(0,0)} = 1$

Hence C.R. equations (14.22) are satisfied at (0,0). Now consider

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}.$$

Substituting $y = mx$ and taking $x \to 0$, we have

$$f'(0) = \lim_{x\to 0} \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}$$

which depends upon m and thus $f'(z)$ does not exist at $z = 0$, though C.R. equations are satisfied at (0,0).

Next, we give the sufficient conditions for a function to be analytic.

Theorem 14.2: ***(Sufficient conditions for a function to be analytic)*** *If the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function $f(z) = u(x, y) + iv(x, y)$ have continuous first order partial derivatives that satisfy the Cauchy-Riemann equations (14.22) at all points in D, then the function $f(z)$ is analytic in D.*

Proof. Let $P(x, y)$ be any fixed point in D and let $Q(x + \Delta x, y + \Delta y)$ be a point in its neighbourhood such that the straight line segment PQ is in D. Then, since u_x, u_y, v_x, v_y exist and are continuous, applying mean value theorem for functions of two variables, we have

$$u(x+\Delta x, y+\Delta y) - u(x, y) = u_x\Delta x + u_y\Delta y + \in_1\Delta x + \in_2\Delta y$$

and

$$v(x+\Delta x, y+\Delta y) - v(x, y) = v_x\Delta x + v_y\Delta y + \in_3\Delta x + \in_4\Delta y,$$

where $\in_1$, $\in_2$, $\in_3$ and $\in_4 \to 0$ as $\Delta z \to 0$. Thus

$$\begin{aligned} f(z+\Delta z) - f(z) &= \{u(x+\Delta x, y+\Delta y) - u(x, y)\} + i\{v(x+\Delta x, y+\Delta y) - v(x, y)\} \\ &= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + (\in_1 + i\in_3)\Delta x + (\in_2 + i\in_4)\Delta y \\ &= (u_x + iv_x)\Delta x + (-v_x + iu_x)\Delta y + (\in_1 + i\in_3)\Delta x + (\in_2 + i\in_4)\,\Delta y, \text{ using (14.22)} \\ &= (u_x + iv_x)(\Delta x + i\Delta y) + (\in_1 + i\in_3)\Delta x + (\in_2 + i\in_4)\Delta y \\ &= (u_x + iv_x)\Delta z + w, \end{aligned}$$

where $w = (\in_1 + i\in_3)\Delta x + (\in_2 + i\in_4)\Delta y$. Hence,

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = u_x + iv_x + \frac{w}{\Delta z}$$

Now, $$\left|\frac{w}{\Delta z}\right| \le (|\in_1| + |\in_3|)\left|\frac{\Delta x}{\Delta z}\right| + (|\in_2| + |\in_4|)\left|\frac{\Delta y}{\Delta z}\right|$$

$$\le |\in_1| + |\in_3| + |\in_2| + |\in_4|, \text{ since } \left|\frac{\Delta x}{\Delta z}\right| \le 1 \text{ and } \left|\frac{\Delta y}{\Delta z}\right| \le 1.$$

Thus, $\displaystyle\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = u_x + iv_x$, that is, $f'(z) = u_x + iv_x$

Therefore $f(z)$ is differentiable at an arbitrary point z in D, that is, $f(z)$ is analytic in D. This proves the sufficiency part, and also we note that in case $f(z)$ is analytic, then $f'(z)$ is given by (14.24) or (14.25).

Example 14.17: Using the Cauchy-Riemann equations, show that

(a) $f(z) = z^3$ is analytic everywhere,

(b) $f(z) = |z|^2$ is analytic nowhere,

(c) $f(z) = \sin z$ is analytic everywhere,

(d) $f(z) = 1/z, z \neq 0$, is analytic at all points except at the point $z = 0$.

Solution: (a) We have, $f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$

Here $u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$

Thus $u_x = 3x^2 - 3y^2, \quad v_x = 6xy, \quad u_y = -6xy, \quad v_y = 3x^2 - 3y^2$

We observe that u_x, u_y, v_x, v_y are continuous everywhere, and

$$u_x = v_y, \; u_y = -v_x$$

at all points.

Hence $f(z) = z^3$ is analytic for every z and further

$$f'(z) = u_x + iv_x = 3x^2 - 3y^2 + i6xy = 3[x^2 + (iy)^2 + 2ixy]$$
$$= 3(x + iy)^2 = 3z^2.$$

(b) We have $f(z) = |z|^2 = z\bar{z} = x^2 + y^2$

Here $u = x^2 + y^2, \qquad v = 0$

Thus $u_x = 2x, \quad v_x = 0, \quad u_y = 2y, \quad v_y = 0$

We observe that u_x, u_y, v_x, v_y are continuous everywhere. Moreover $u_x = v_y$ is satisfied at all points on $x = 0$, (the imaginary axis) and $u_y = -v_x$ is satisfied at all points on $y = 0$, (the real axis) so both Cauchy-Riemann equations are satisfied only at (0, 0). Thus $f(z) = |z|^2$ is differentiable only at (0, 0) and hence it is analytic nowhere.

(c) We have, $f(z) = \sin z = \sin (x + iy) = \sin x \cosh y + i \cos x \sinh y$

Here $u = \sin x \cosh y \quad v = \cos x \sinh y$

Thus $u_x = \cos x \cosh y \quad v_x = -\sin x \sinh y$

$u_y = -\sin x \sinh y \quad v_y = \cos x \cosh y$

We observe that u_x, u_y, v_x, v_y are continuous everywhere and $u_x = v_y, \quad u_y = -v_x$ at all points. Hence $f(z) = \sin z$ is analytic everywhere.

Further $f'(z) = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y$

$$= \cos x \cos (iy) - \sin x \sin (iy) = \cos (x + iy) = \cos z.$$

(d) We have, $$f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Here $$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}$$

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad v_x = \frac{2xy}{(x^2 + y^2)^2} \quad u_y = \frac{-2xy}{(x^2 + y^2)^2}, \quad v_y = \frac{-(x^2 - y^2)}{(x^2 + y^2)^2}$$

We observe that u_x, u_y, v_x, v_y are continuous everywhere except at the point $z = 0$, and $u_x = v_y$, $u_y = -v_x$ at all points except at $z = 0$. Hence $f(z) = 1/z$ is analytic everywhere except at $z = 0$. Further,

$$f'(z) = u_x + iv_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{2ixy}{(x^2 + y^2)^2}$$

$$= \frac{-[x^2 + (iy)^2 - 2ixy]}{(x^2 + y^2)^2} = \frac{-(x - iy)^2}{(x^2 + y^2)^2} = \frac{-(\bar{z})^2}{(z\bar{z})^2} = -\frac{1}{z^2}, z \neq 0$$

Example 14.18: Show that the function $f(z) = \sqrt{xy}$ is not analytic at the origin even though Cauchy-Reimann equations are satisfied there.

Solution: We have $f(z) = \sqrt{xy}$. Thus $u = \sqrt{xy}$ and $v = 0$

By definition $\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$

Similarly, $\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = 0$, $\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = 0$, and $\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = 0$

Hence C.R. equations are satisfied at the origin. But

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{y = mx \\ x \to 0}} \frac{\sqrt{xy} - 0}{x + iy} = \lim_{x \to 0} \frac{x\sqrt{m}}{x(1 + im)} = \frac{\sqrt{m}}{(1 + im)}$$

depends on m and hence is not unique.

Thus $f'(0)$ does not exist and hence function is not analytic at (0, 0).

Example 14.19: If $f(z) = u + iv$ is an analytic function of z, and $u - v = (x - y)(x^2 + 4xy + y^2)$, then find $f(z)$.

Solution: We have, $u - v = (x - y)(x^2 + 4xy + y^2)$

Differentiating it partially w.r.t. x and y we have, respectively

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = (x^2 + 4xy + y^2) + (x - y)(2x + 4y)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$$

Adding these two equations and using the C.R. equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we obtain

$$2\frac{\partial u}{\partial y} = 6(x^2 - y^2), \quad \text{or} \quad \frac{\partial u}{\partial y} = 3(x^2 - y^2) \qquad ...(14.26)$$

Integrating w.r.t. y, we have

$$u = 3x^2y - y^3 + c_1(x), \qquad ...(14.27)$$

where $c_1(x)$ is arbitrary function of x only.

Rewriting (14.26) using C.R. equations, as

$$-\frac{\partial v}{\partial x} = 3(x^2 - y^2), \quad \text{or} \quad \frac{\partial v}{\partial x} = 3(y^2 - x^2)$$

Integrating w.r.t. x, we have

$$v = 3y^2x - x^3 + c_2(y), \qquad ...(14.28)$$

where $c_2(y)$ is an arbitrary function of y only.

From (14.27) and (14.28) we have

$$u - v = 3xy(x - y) + (x^3 - y^3) + c_1(x) - c_2(y)$$

Comparing it with the given expression, we obtain

$$c_1(x) - c_2(y) = 0, \quad \text{or} \quad c_1(x) = c_2(y) = \text{constant, say } a.$$

Hence, $$f(z) = 3x^2y - y^3 + i(3y^2x - x^3) + A,$$

where $A = a + ia$ is a complex constant.

Example 14.20: If $f(z)$ is an analytic function of z, then show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

Solution: Let $f(z) = u + iv$, then $|f(z)| = (u^2 + v^2)^{\frac{1}{2}}$

Consider $$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}(u^2 + v^2)^{\frac{1}{2}} = \frac{1}{2}(u^2 + v^2)^{-\frac{1}{2}} \cdot \left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)$$

$$= \frac{u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}}{(u^2 + v^2)^{\frac{1}{2}}}$$

or, $$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = \frac{\left[u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}\right]^2}{u^2 + v^2} \qquad ...(14.29)$$

Similarly, $$\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left[u\dfrac{\partial u}{\partial y}+v\dfrac{\partial v}{\partial y}\right]^2}{u^2+v^2}.$$

Since $f(z)$ is analytic, using C.R. equations, it becomes

$$\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left\{v\dfrac{\partial u}{\partial x}-u\dfrac{\partial v}{\partial x}\right\}^2}{u^2+v^2} \qquad ...(14.30)$$

From (14.29) and (14.30), we have

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{u^2\left(\dfrac{\partial u}{\partial x}\right)^2+v^2\left(\dfrac{\partial v}{\partial x}\right)^2+u^2\left(\dfrac{\partial v}{\partial x}\right)^2+v^2\left(\dfrac{\partial u}{\partial x}\right)^2}{u^2+v^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2$$

14.4.3 Polar Form of the Cauchy-Riemann Equations

Sometimes it is convenient to express $f(z)$ in terms of the polar coordinates (r, θ) as $f(z) = u(r, \theta) + iv(r, \theta)$, where $z = re^{i\theta}$, which gives

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

By chain rule of differentiation, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial u}{\partial x}+(\sin\theta)\frac{\partial u}{\partial y} \qquad ...(14.31)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial u}{\partial x}+(r\cos\theta)\frac{\partial u}{\partial y} \qquad ...(14.32)$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial v}{\partial x}+(\sin\theta)\frac{\partial v}{\partial y} \qquad ...(14.33)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial v}{\partial x}+(r\cos\theta)\frac{\partial v}{\partial y} \qquad ...(14.34)$$

Using C-R equations, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ in (14.33), we have

$$\frac{\partial v}{\partial r} = -(\cos\theta)\frac{\partial u}{\partial y} + (\sin\theta)\frac{\partial u}{\partial x} = -\frac{1}{r}\left[(-r\sin\theta)\frac{\partial u}{\partial x} + (r\cos\theta)\frac{\partial u}{\partial y}\right] = -\frac{1}{r}\frac{\partial u}{\partial \theta}, \quad \text{using (14.32).}$$

Similarly from (14.34), we have

$$\frac{\partial v}{\partial \theta} = (-r\sin\theta)\left(-\frac{\partial u}{\partial y}\right) + (r\cos\theta)\frac{\partial u}{\partial x} = r\left[(\cos\theta)\frac{\partial u}{\partial x} + (\sin\theta)\frac{\partial u}{\partial y}\right] = r\frac{\partial u}{\partial r}, \text{ using (14.31)}$$

Therefore, the C.R. equations in polar coordinates are

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \textit{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \qquad \text{...(14.35)}$$

Further, $f(z) = u + iv$, gives

$$f'(z).e^{i\theta} = u_r + iv_r, \quad (\text{using } z = re^{i\theta})$$

or,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$$

$$= e^{-i\theta}\left(u_r - \frac{i}{r}u_\theta\right)$$

$$= e^{-i\theta}\left(\frac{1}{r}v_\theta + iv_r\right)$$

using C.R. equations in polar coordinates.

All these are the various expressions for the derivatives of $f(z)$ in terms of the polar coordinates.

14.5 HARMONIC FUNCTIONS. LAPLACE EQUATION

A real valued function $\phi(x, y)$ of two variables x and y that has continuous second order partial derivatives and satisfy the Laplace equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \text{...(14.36)}$$

is called a harmonic function.

The Laplace equation (14.36) is of great practical importance and occurs frequently in the study of fluid flow, heat conduction, gravitation and electrostatic and the *real and imaginary parts of an analytic function are harmonic functions and thus satisfy the Laplace equation.* This is one of the main reasons for the importance of complex analysis in engineering mathematics.

We have the following result:

Theorem 14.3: *If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v satisfy the Laplace equation (14.36) in D and have continuous second order partial derivatives in D.*

Proof. Since $f(z) = u + iv$ is analytic in D, thus $u_x = v_y$ and $u_y = -v_x$.

Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y, we get

$$u_{xx} = v_{yx} \text{ and } u_{yy} = -v_{xy} \qquad ...(14.37)$$

Since u and v have continuous partial derivatives of all orders, so mixed second order derivatives are equal, that is $v_{yx} = v_{xy}$.

Thus, from (14.37), we obtain $u_{xx} + u_{yy} = 0$

Similarly, differentiating $v_y = u_x$ with respect to y and $v_x = -u_y$ with respect to x, we get $v_{yy} = u_{xy}$ and $v_{xx} = -u_{yx}$

Adding these two, we get $v_{xx} + v_{yy} = 0$.

Thus, if $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its real and imaginary parts are harmonic functions in D.

Also we observe that, if $f(z) = u(x,y) + iv(x, y)$ is analytic in D, then the harmonic functions u and v are a 'related pair', since they satisfy the Cauchy-Reimann equations. The functions $u(x, y)$ and $v(x, y)$ are called the *conjugate harmonic functions* of each other in D. Given one harmonic function, the conjugate harmonic function can be obtained by using the Cauchy-Reimann equations. *A conjugate of a given harmonic function is uniquely determined upto an arbitrary real additive constant.*

Remark. If $u(x, y)$ and $v(x, y)$ are any two harmonic functions in D, then $f(z) = u + iv$ need not be analytic in D. For example, $u = \sinh x \cos y$ and $v = \cosh x \cos y$ are harmonic functions but $f(z) = u + iv$ is not an analytic function since it does not satisfy the C.R. equations.

Next, we give another interesting result concerning the analytic functions.

Theorem 14.4: *If $f(z) = u + iv$ is an analytic function, then the two families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system.*

Proof. Differentiating $u(x, y) = c_1$ partially w.r.t. x, we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial y}{\partial v/\partial x} = m_1, \quad \text{say}$$

Similarly, differentiating $v(x, y) = c_2$ partially w.r.t. x, we obtain

$$\frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2, \quad \text{say}$$

Here $m_1 m_2 = -1$, thus $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system of curves.

14.5.1 Polar Form of the Laplace Equation

In polar system the real and imaginary parts of an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ satisfy the Laplace equation in the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \text{ and } \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

respectively.

Functions $u(r, \theta)$ and $v(r, \theta)$ satisfy the C.R. Eqs. (14.35), given by

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

Consider the equation, $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$. Differentiating w.r.t. r, we get

$$\frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} \qquad ...(14.38)$$

Similarly considering, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$, and differentiating w.r.t. θ, we get

$$\frac{\partial^2 v}{\partial \theta\,\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \qquad ...(14.39)$$

Assuming that $v(r, \theta)$ has continuous second order partial derivatives, so that $\frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial^2 v}{\partial \theta\,\partial r}$.

Thus from (14.38) and (14.39), we obtain

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(14.40)$$

On similar lines the corresponding equation for $v(r, \theta)$ can be obtained.

The real valued polar functions $u(r, \theta)$ of two variables r, θ that has continuous second order partial derivatives and satisfy the Laplace Eq. (14.40) are called 'harmonic functions'.

Example 14.21: If $f(z)$ is an analytic function with constant modulus, then $f(z)$ is constant.

Solution: Let $f(z) = u + iv$, then $|f(z)| = \sqrt{u^2 + v^2} = \text{constant}$

Differentiating $u^2 + v^2 = \text{constant}$ w.r. t. x, we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$$

Differentiating it again w.r.t. x, we have,

$$\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} = 0$$

Similarly, $$\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} = 0$$

Adding these two and using $\nabla^2 u = 0$ and $\nabla^2 v = 0$, (since u and v are harmonic, being real and imaginary parts of an analytic function $f(z) = u + iv$), we obtain

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = 0,$$

which implies,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$$

Thus u and v are independent of both x and y and hence $f(z)$ is constant.

Example 14.22: If $f(z)$ is an analytic function of z, then prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\operatorname{Re} f(z)|^2 = 2\,|f'(z)|^2.$$

Solution: Let $f(z) = u + iv$, then $\operatorname{Re} f(z) = u$. Consider

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\operatorname{Re} f(z)|^2 = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)$$

$$= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2}$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\nabla^2 u$$

$$= 2\,|f'(z)|^2,$$

since $f'(z) = u_x - iu_y$ and $\nabla^2 u = 0$, u being harmonic function.

Example 14.23: Show that the function $v(x, y) = \ln(x^2 + y^2) + x - 2y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$.

Solution: We have $v(x, y) = \ln(x^2 + y^2) + x - 2y$. This gives

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1, \quad \frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2,$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}, \quad \frac{\partial^2 v}{\partial y^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Since $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$, thus the function $v(x, y)$ is harmonic.

Example 14.26: Two concentric circular cylinders of radii a and b, $(a < b)$, are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = A \ln z + c$, prove that the capacitance per unit length of the capacitor formed by them in vacuum is $2\pi/\ln (b/a)$.

Solution: Let $w = \phi + i\psi$, where ϕ is the potential function and ψ is the corresponding flux function.

Thus, $$\phi + i\psi = A \ln z + c$$
$$= (A \ln r + c) + i A\theta$$

Therefore, $$\phi = A \ln r + c \text{ and } \psi = A\theta$$

So that $$\phi_1 = A \ln a + c \text{ and } \phi_2 = A \ln b + c,$$

and thus the potential difference $\phi_2 - \phi_1 = A \ln (b/a)$

Also the total charge Q is given by

$$Q = \int_0^{2\pi} d\psi = A \int_0^{2\pi} d\theta = 2\pi A.$$

Hence the capacitance, that is, the charge required to maintain a unit potential difference

$$= \frac{Q}{\phi_2 - \phi_1} = \frac{2\pi A}{A \ln (b/a)} = \frac{2\pi}{\ln (b/a)}$$

Example 14.27: Show that the function $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

Solution: We have, $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. This gives

$$v_r = 2r \cos 2\theta - \cos \theta, \qquad v_{rr} = 2 \cos 2\theta$$

and $$v_\theta = -2r^2 \sin 2\theta + r \sin \theta \qquad v_{\theta\theta} = -4r^2 \cos 2\theta + r \cos \theta$$

Since, $v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 2 \cos 2\theta + 2 \cos 2\theta - \frac{1}{r} \cos \theta - 4 \cos 2\theta + \frac{1}{r} \cos \theta = 0$, thus the function $v(r, \theta)$ satisfies the Laplace equation and, therefore, is harmonic.

From the Cauchy-Riemann equations in polar coordinates, refer (14.35), $ru_r = v_\theta = -2r^2 \sin 2\theta + r \sin \theta$, we have

$$u_r = -2r \sin 2\theta + \sin \theta \qquad \text{...(14.43)}$$

Similarly, $-\frac{1}{r} u_\theta = v_r = 2r \cos 2\theta - \cos \theta$, gives

$$u_\theta = -2r^2 \cos 2\theta + r \cos \theta \qquad \text{...(14.44)}$$

Integrating (14.43) with respect to r, we get

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta),$$

where $\phi(\theta)$ is an arbitrary function of θ.

Differentiating it partially w.r.t. θ, we have

$$u_\theta = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \qquad \text{...(14.45)}$$

From (14.44) and (14.45), we get $\phi'(\theta) = 0$, or $\phi(\theta) = c$

Thus, $u = -r^2 \sin 2\theta + r \sin \theta + c$

Hence,
$$\begin{aligned} f(z) = u + iv &= (-r^2 \sin 2\theta + r \sin \theta + c) + i(r^2 \cos 2\theta - r\cos\theta + 2) \\ &= r^2(-\sin 2\theta + i\cos 2\theta) + r(\sin\theta - i\cos\theta) + c + 2i \\ &= i(r^2 e^{2i\theta} - re^{i\theta}) + c + 2i \end{aligned}$$

EXERCISE 14.3

1. Determine which of the following functions are analytic:

(a) z^6 (b) $z\bar{z}$ (c) $2xy + i(x^2 - y^2)$

(d) $\dfrac{(x - iy)}{x^2 + y^2}$ (e) $Re z / Im z$ (f) $Re(z^3)$

2. Show that

(a) $f(z) = xy + iy$ is everywhere continuous but is not analytic.

(b) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

3. Prove that an analytic function $f(z)$ with $Re . f(z) =$ const. is constant.

4. Determine whether the following functions are harmonic. If answer is yes, find a corresponding analytic function $f(z) = u + iv$

(a) $u = x/(x^2 + y^2)$ (b) $u = -e^{-x} \sin y$

(c) $u = e^{2x}(x \cos 2y - y \sin 2y)$ (d) $u = x \sin x \cosh y - y \cos x \sinh y$

(e) $v = e^{-x}(x \sin y - y \cos y)$ (f) $u = \sin 2x / (\cosh 2y - \cos 2x)$

5. Determine the analytic function $f(z) = u + iv$, if

(a) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$, $f(\pi/2) = 0$

(b) $u - v = e^{-x}[(x - y)\sin y - (x + y)\cos y]$, $f(0) = 0$

6. Determine whether the following functions are harmonic. If yes, find a corresponding analytic function $f(z) = u(r, \theta) + iv(r, \theta)$

(a) $v(r, \theta) = r^2 \cos 2\theta$ (b) $u(r, \theta) = \left(r + \dfrac{1}{r}\right)\cos\theta$, $r \neq 0$

7. If u is a harmonic function, then show that u^2 is not a harmonic function unless u is a constant.

8. If $f(z)$ is an analytic function of z, then show that
$$\nabla^2[|f(z)|^p] = p^2 |f(z)|^{p-2} |f'(z)|^2,$$
where ∇^2 is the Laplacian operator and p is a real greater than 1.

9. If $f(z)$ is an analytic function of z, prove that

(a) $$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

(b) $$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\} \ln|f(z)| = 0$$

10. Consider the analytic function $f(z) = e^z$, $z \neq 0$. Find its level curves. Show that these curves are mutually orthogonal.

14.6 GEOMETRIC ASPECTS OF ANALYTIC FUNCTIONS

So far in this chapter we have discussed the analytical aspects, such as the values of the function, differentially and analyticity of the function of a complex variable. We have observed that a function $w = f(z)$ is actually a mapping from a given region in the z-plane to a corresponding one in the w-plane. To get a better understanding of the properties of function $f(z)$, in this section we pay our attention to the geometrical aspects of the mapping $w = f(z)$.

A relationship $u = f(x, y, t)$ containing three independent real variables x, y and t requires a four-dimensional space for geometric representation. Similar difficulties arise when one attempts to represent graphically complex function $w = f(z) = u(x, y) + iv(x, y)$, with $z = x + iy$. For to each pair of values (x, y), there correspond two values (u, v) and in order to plot a quadruplet of real values (u, v, x, y) we need a four-dimensional space.

However, we visualize $w = f(z)$ as a relationship which assigns to each point z in its domain of definition D, the corresponding point $w = f(z)$ in the w-plane. Thus we need two planes, the z-plane in which we plot values of z and the w-plane in which we plot the corresponding value $w = f(z)$. We say that f defines a mapping of region D in the $z = x + iy$ plane to another region D' defined by $w = f(z)$ in the $w = u + iv$ plane. On separating $w = f(z)$ into real and imaginary parts, we obtain two real functions

$$u = u(x, y), \quad v = v(x, y)$$

which can be viewed as the equations of a transformation that maps a specified set of points (x, y) in the xy-plane into another set of points (u, v) in the uv-plane. We adopt this mode of studying complex functions and introduce a few standard transformations in this section.

14.6.1 Translation: $w = z + c$

Set $z = x + iy$, $w = u + iv$ and $c = h + ik$ in $w = z + c$, we get

$$u + iv = x + iy + h + ik = (x + h) + i(y + k)$$

Hence, $\quad u = x + h$ and $v = y + k$

Thus under translation the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + h, y + k)$ in the w-plane. Thus this transformation maps a region in the z-plane into a region in the w-plane of the same shape and size, where each point is moved h units in the direction of the x-axis and k units in the direction of y-axis.

For example, the rectangle $OABC$ with the vertices $O(0, 0)$, $A(2, 0)$, $B(2, 1)$ and $C(0, 1)$ in the z-plane, refer Fig. 14.8a, is transformed to rectangle $O'A'B'C'$ with vertices $O'(1, 2)$, $A'(3, 2)$, $B'(3, 3)$ and $C'(1, 3)$ in the w-plane under the transformation $w = z + (1 + 2i)$, refer Fig. 14.8b.

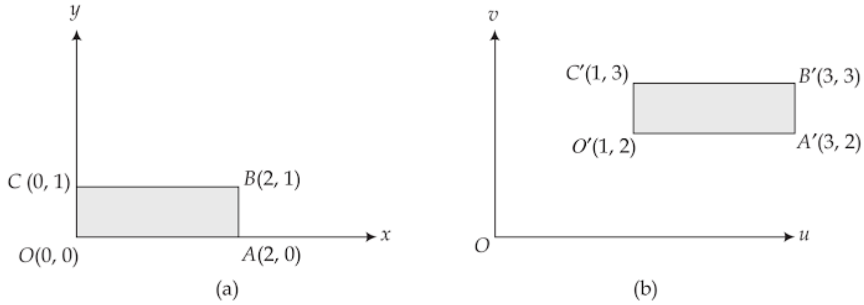


Fig. 14.8

14.6.2 Scaling and Rotation: $w = az$

To study this transformation it is convenient to use polar-coordinates.

Set $z = re^{i\theta}$, $w = \rho e^{i\phi}$ and $a = Ae^{i\alpha}$, $w = az$ becomes,

$$\rho e^{i\phi} = Are^{i(\alpha + \theta)}$$

so that, $$\rho = Ar,\ \phi = \alpha + \theta.$$

Thus the point $P(r, \theta)$ in the z-plane is mapped onto the point, P' $(Ar, \alpha + \theta)$ in the w-plane. Hence the transformation results in *magnification*, if $|a| = A > 1$, or *contraction*, if $|a| = A < 1$, accompanied by a rotation through an angle α. Thus a square in the z-plane is transformed into a square, a circle of radius R is transformed into a circle of radius AR. If $A = 1$, we have a pure rotation through an angle α.

The same conclusion can be reached by setting $w = u + iv$, $z = x + iy$ and $a = h + ik$ in $w = az$ and deriving the transformation equations $u = hx - ky$, $v = kx + hy$ in cartesian co-ordinates.

For example, a rectangular region $ABCD$ with vertices at $A(2, 1)$, $B(3, 1)$, $C(3, 3)$ and $D(2, 3)$ in the z-plane, refer Fig. 14.9a, is transformed into the rectangle $A'B'C'D'$, with vertices at $A'(1/\sqrt{2}, 3/\sqrt{2})$, $B'(2/\sqrt{2}, 4/\sqrt{2})$, $C'(0, 6/\sqrt{2})$ and $D'(-1/\sqrt{2}, 5/\sqrt{2})$ in the w-plane under the transformation $w = (1 + i)z/\sqrt{2}$, as shown in Fig. 14.9b.

The rectangular region $ABCD$ has been rotated by an angle $\pi/4$ but the length of the sides remains unchanged since $|w| = 1$.

14.6.3 Inversion and Reflection: $w = \dfrac{1}{z}$, $z \neq 0$

To study this relationship we again use polar co-ordinates. On setting $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, we get

$$\rho e^{i\phi} = \left(\frac{1}{r}\right)e^{-i\theta}, \text{ so that } \rho = \frac{1}{r} \text{ and } \phi = -\theta$$

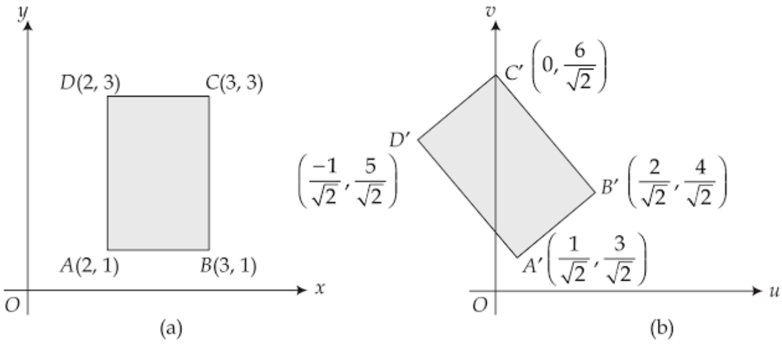


Fig. 14.9

Thus the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(1/r, -\theta)$ in the w-plane. The unit circle $|z| = 1$ is transformed into the unit circle $|w| = 1$ in the w-plane. As the point P traces out the circle $|z| = 1$ in the clockwise direction the corresponding point P' in the w-plane traces out the circle $|w| = 1$ in the counter-clockwise direction. Points in the interior of $|z| = 1$ are mapped into points in the exterior of $|w| = 1$, and the exterior of $|z| = 1$ is mapped into the interior of $|w| = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and if we consider the inverse transformation $z = 1/w$, we observe that $w = 0$ corresponds to $z = \infty$.

The equations of transformation defined by $w = 1/z$ in cartesian co-ordinates have the form

$$u = \frac{x}{x^2 + y^2}, \; v = -\frac{y}{x^2 + y^2},$$

with the inverse transformation equations as

$$x = \frac{u}{u^2 + v^2}, \; y = -\frac{v}{u^2 + v^2} \qquad \text{...(14.46)}$$

Next, we state an important result concerning the inversion transformation:

The inversion transformation $w = 1/z$ maps a circle onto a circle or to a straight line if the circle passes through the origin.

To prove it consider the general equation of a circle in the z-plane given by

$$x^2 + y^2 + 2gx + 2fy + c + 0 \qquad \text{...(14.47)}$$

Using (14.46), (14.47) becomes

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + 2g\frac{u}{u^2 + v^2} + 2f\frac{-v}{u^2 + v^2} + c = 0$$

or, $$c(u^2 + v^2) + 2gu - 2fv + 1 = 0, \qquad \text{...(14.48)}$$

which is the equation of a circle in the w-plane. If $c = 0$, the circle (14.47) passes through the origin and its image (14.48) reduces to the straight line $2gu - 2fv + 1 = 0$.

14.6.4 Bilinear Transformation

The transformation

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0, \qquad \text{...(14.49)}$$

where a, b, c and d are complex constants, is called the *bilinear transformation*, or '*Mölius transformation*', or '*linear fractional transformation*'. The condition $ad - bc \neq 0$ ensures that $w(z)$ is not merely a constant or 0/0.

Since $w' = \dfrac{ad-bc}{(cz+d)^2}$ exists for all z except $z = -\dfrac{d}{c}$, we observe that the bilinear transformation is *conformal*, (refer Section (14.7)) everywhere except at $z = -d/c$, a point at which w is also not defined. The point $z = -d/c$ is called the '*pole*' of the transformation and the image of this point can be regarded as the *'point at infinity'* in the w-plane denoted as $w = \infty$. The resultant w-plane is called the *extended complex plane*.

The corresponding inverse mapping is

$$z = -\frac{dw-b}{cw-a}, \quad w \neq \frac{a}{c}$$

which gives a unique z for each w. Hence besides being conformal, the bilinear transformation is also *one-to-one* on the extended z and w-planes.

Next if $a \neq 0$, $c \neq 0$ then we can rewrite (14.49) as

$$w = \frac{a}{c} + \frac{bc-ad}{c^2}\left[\frac{1}{z+(d/c)}\right],$$

which can be considered a composition of following mappings

(a) Translation: $w_1 = z + d/c$

(b) Inversion: $w_2 = 1/w_1$

(c) Scaling and rotation: $w_3 = \dfrac{bc-ad}{c^2} w_2$; and then again

(d) Translation: $w_4 = \dfrac{a}{c} + w_3$

Therefore the general bilinear transformation can be considered a composition of the translation, inversion, scaling and rotation, and since under each of these the totatily of the circles and straight lines in the z-plane are mapped as the totality of circles and straight lines in the w-plane, thus we have the following theorem:

Theorem 14.5: *Every bilinear transformation of the form (14.49) maps every circle or straight line in the z-plane onto either a circle or a straight line in the w-plane.*

14.6.5 Fixed Points or Invariant Points

Fixed points of a mapping $w = f(z)$ are the points that are mapped onto themselves under the mapping, thus they are obtained from $w = f(z) = z$.

For example, the identity mapping $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has points on the whole of the real axis as fixed points while the inversion mapping $w = 1/z$ has two fixed points $z = \pm 1$.

For the bilinear transformation (14.49), fixed points are given by

$$z = (az + b)/(cz + d), \text{ or } \quad cz^2 + (d - a)z - b = 0$$

This is quadratic in z whose coefficients all vanish if, and only if the mapping is the indentity mapping $w = z$ and in this case, $a = d \neq 0$, $b = c = 0$. Hence we have the following theorem:

Theorem 14.6: *A bilinear transformation, not the identity has at most two fixed points. If it is known to have three or more fixed points it must be the identity mapping $w = z$.*

14.6.6 Procedure to Find Bilinear Transformations

It appears that a bilinear transformation of the from (14.49) is determined by the four constants a, b, c, d, but by dividing the numerator and the denominator on the right of (14.49) by one of the four constants, it is clear that one constant can be dropped and thus the three conditions determine a unique bilinear transformation. Thus we have the following theorem:

Theorem 14.6: *Three given distinct points z_1, z_2, z_3 can always be mapped onto three prescribed distinct points w_1, w_2, w_3 by one, and only one bilinear transformation and this transformation is given by*

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

If one of these points is the point at infinity, the quotient of the two differences containing these points must be replaced by 1.

Proof. Let the required transformation be $w = \dfrac{az + b}{cz + d}$, $\quad ad - bc \neq 0$

It is given that $\quad w_j = \dfrac{az_j + b}{cz_j + d}$, $\quad j = 1, 2, 3$, $z_i \neq z_j$, for $i \neq j$.

For $j = 1$, $\quad w - w_1 = \dfrac{az + b}{cz + d} - \dfrac{az_1 + b}{cz_1 + d} = \dfrac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)}$

Similarly, $\quad w - w_2 = \dfrac{(ad - bc)(z - z_2)}{(cz + d)(cz_2 + d)}$

Also, $\quad w_3 - w_1 = \dfrac{(ad - bc)(z_3 - z_1)}{(cz_3 + d)(cz_1 + d)}$ and $\quad w_3 - w_2 = \dfrac{(ad - bc)(z_3 - z_2)}{(cz_3 + d)(cz_2 + d)}$

Hence, we have

$$\frac{w - w_1}{w - w_2} = \frac{(z - z_1)(cz_2 + d)}{(z - z_2)(cz_1 + d)} \text{ and } \frac{w_3 - w_1}{w_3 - w_2} = \frac{(z_3 - z_1)(cz_2 + d)}{(z_3 - z_2)(cz_1 + d)}$$

Thus,
$$\frac{w - w_1}{w - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1} \qquad \text{...(14.50)}$$

This proves the theorem.

If one of the point is infinity, say $w_1 = \infty$, then

$$\lim_{w_1 \to \infty} \frac{w - w_1}{w_3 - w_1} = \lim_{w_1 \to \infty} \left[\frac{(w/w_1) - 1}{(w_3/w_1) - 1} \right] = \frac{-1}{-1} = 1$$

For example, the bilinear transformation that maps $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ onto $w_1 = -1$, $w_2 = -i$, $w_3 = 1$ respectively, is obtained from (14.50) by substituting the respective values. We obtain,

$$\frac{w - (-1)}{w - (-i)} \cdot \frac{1 - (-i)}{1 - (-1)} = \frac{z - 0}{z - 1} \cdot \frac{\infty - 1}{\infty - 0}$$

Replacing $\dfrac{\infty - 1}{\infty}$ by 1 and simplifying, we get $w = (z - i)/(z + i)$ as the desired mapping.

Remark. The property (14.50) is also stated as that '*a bilinear transformation preserves cross ratio of four points*.'

Example 14.28: Find the image of the infinite strip $1/4 \le y \le 1/2$ under the transformation $w = 1/z$.

Solution: If $z = x + iy$, $w = u + iv$, then under the given transformation $w = 1/z$, we have

$$x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

Thus, $y = \dfrac{1}{4}$ gives $\dfrac{-v}{u^2 + v^2} = \dfrac{1}{4}$, that is, $u^2 + v^2 + 4v = 0$, or $u^2 + (v + 2)^2 - 4 = 0$,

and $y = \dfrac{1}{2}$ gives $\dfrac{-v}{u^2 + v^2} = \dfrac{1}{2}$, that is, $u^2 + v^2 + 2v = 0$, or $u^2 + (v + 1)^2 - 1 = 0$.

Hence the infinite strip $1/4 \le y \le 1/2$ in the z-plane, as shown in Fig. 14.10a, is transformed into the region $u^2 + (v + 2)^2 - 4 \le y \le u^2 + (v + 1)^2 - 1$ in the w-plane, that is, the region between the two circles both passing through (0, 0). Circles are: $u^2 + (v + 2)^2 = 4$ with center (0, –2) and radius 2 and, $u^2 + (v + 1)^2 = 1$ with center (0, –1) and radius 1 as shown in Fig. 14.10b.

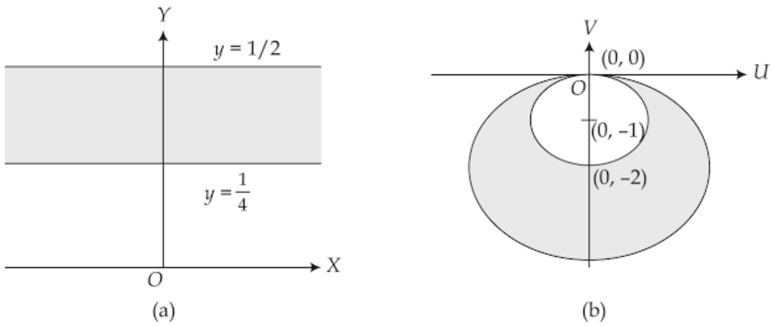


Fig. 14.10

Example 14.29: Find the image of the region $|z - i| < 2$ under the mapping $w = (1 + i)/(z + i)$.

Solution: The given mapping is

$$w = \frac{1+i}{z+i}, \text{ or } z = \frac{1+i(1-w)}{w}$$

Hence, $$|z - i| = \left|\frac{1+i(1-w)}{w} - i\right| = \left|\frac{1+i(1-2w)}{w}\right|$$

Thus, $$|z - i| < 2 \text{ gives } |1 + i(1 - 2w)| < 2|w|$$

or, $$|(1 + 2v) + i(1 - 2u)| < 2\,|u + iv|, \text{ where } w = u + iv$$

or, $$(1 + 2v)^2 + (1 - 2u)^2 < 4(u^2 + v^2)$$

or, $$2v - 2u + 1 < 0$$

Therefore, the interior of the circular disk $|z - i| < 2$ as shown in Fig. 14.11a is mapped as the half plane below the line $2v - 2u + 1 = 0$ and the boundary circle $|z - i| = 2$ is mapped as the line $2v - 2u + 1 = 0$, as shown in the Fig. 14.11b.

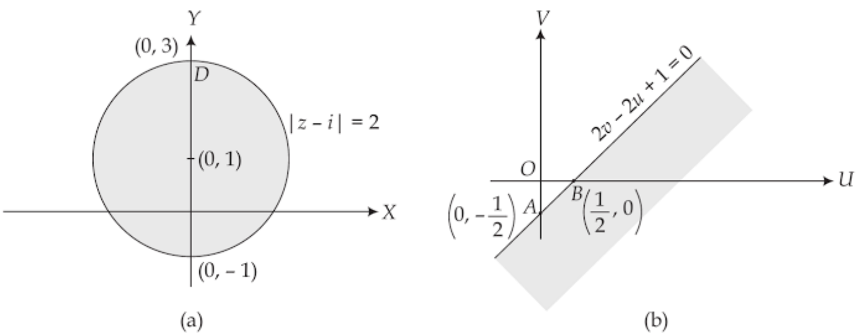


Fig. 14.11

Example 14.30: Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence, find the image of $|z| < 1$.

Solution: Let the required bilinear transformation be $w = \dfrac{az+b}{cz+d}$

Substituting the corresponding values of w and z, we get

$$i = \frac{a+b}{c+d}, \quad 0 = \frac{ai+b}{ci+d}, \quad -i = \frac{a(-1)+b}{c(-1)+d},$$

which give respectively

$$(a + b) - i(c + d) = 0, \quad b + ia = 0, \text{ and } (-a + b) + i(-c + d) = 0$$

Solving these equations in term of a, we get $b = -ia$, $c = -a$, $d = -ia$.

Substituting for b, c and d, the required transformation becomes

$$w = \frac{az - ia}{-az - ia}, \text{ or } w = \frac{i-z}{i+z}, \text{ or } z = \frac{i(1-w)}{1+w}$$

Thus the region $|z| < 1$ is mapped onto the region

$$\left|\frac{i(1-w)}{1+w}\right| < 1, \text{ or } \frac{|i||1-w|}{|1+w|} < 1, \text{ or } |1 - w| < |1 + w|$$

or, $\quad |1 - u - iv| < |1 + u + iv|$, where $w = u + iv$

or, $\quad (1 - u)^2 + v^2 < (1 + u)^2 + v^2$, or $u > 0$

Hence, the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped into the entire positive half plane $u > 0$ in the w-plane. Further we note that the circle $|z| = 1$ in the z-plane is mapped into the imaginary axis $u = 0$ in the w-plane.

EXERCISE 14.4

1. Find the image of the triangle with vertices at i, $1 + i$, $1 - i$ in the z-plane under the transformation $w = 3z + 4 - 2i$.
2. Find the image of the rectangular region with vertices at (0, 0), (1, 0) (1, 2) and (0, 2) under the transformation $w = \sqrt{2}\, e^{-i\pi/4}\,[z + (1 - i)]$.
3. Find the image of $|z - 2i| = 2$ under the transformation $w = 1/z$.
4. Show that every circle in the z-plane maps by the transformation $w = 1/z$ into a circle in the w-plane if one cansiders straight lines as the limiting cases of circles.
5. Find the invariant points of the transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.
6. Show that if a is complex with $|a| < 1$, then $w = (z - a)/(1 - \bar{a}z)$ maps $|z| < 1$ onto $|w| < 1$, with 'a' being mapped to the origin.

7. Find the bilinear transformation which maps
   (a) The points $z = 1, i, -1$ into the points $w = 0, 1, \infty$
   (b) The points $z = 0, -1, \infty$ into the points $w = -1, -2 - i, i$
   (c) $\text{Re}(z) > 0$ into interior of unit circle so that $z = \infty, i, 0$ map into $w = -1, -i, 1$.
8. Find the image of the closed half disk $|z| \leq 1$, $\text{Im}(z) \geq 0$ under the bilinear transformation $w = z/(z + 1)$.
9. Find all bilinear transformations whose fixed points are
   (a) -1 and 1 (b) i and $-i$
10. Show that the image of the half plane $x + y > 0$ under the bilinear transformation $w = (z - 1)/(z + i)$ is the interior of the unit disk $|w| = 1$.

14.7 CONFORMAL MAPPING

In this section we consider the most important geometrical property of the mappings defined by analytic functions, namely, *the conformality* or *the angle preserving property*. Also, we shall discuss a few specific conformal mappings like z^2, e^z, $z + 1/z$, $\sin z$, $\cos z$, $\cosh z$ etc.

14.7.1 The Conformal Mapping

A mapping in the plane is said to be 'angle-preserving', or 'conformal', if it preserves angles between oriented curves in magnitude as well as in sense.

The angle between two oriented curves is the angle α, $(0 \leq \alpha \leq \pi)$, between their oriented tangents at their point of intersection, as shown in Fig. 14.12.

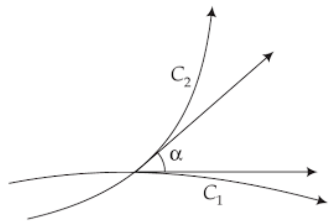


Fig. 14.12

Angle preserving property of analytic functions. We have the following result.

Theorem 14.7: *The mapping defined by an analytic function $f(z)$ is conformal except at points where the derivative $f'(z) = 0$, called the critical points.*

Proof. Let $w = f(z)$ be an analytic function and defines a mapping from a region D in the z-plane onto a region D' in the w-plane and let C be a continuous curve in the z-plane passing through a point $z_0 \in D$, with the parametric representation

$$z(t) = x(t) + iy(t), \quad a \leq t \leq b$$

The increasing value of t is taken as the positive direction of the curve C. We assume $z(t)$ differentiable, $\dot{z}(t) \neq 0$ and continuous on C. Therefore C is a smooth curve as shown in Fig. 14.13a.

Let $C^{\bullet}$ be the image curve of C in the w-plane as shown in Fig. 14.13b. Then,

$$w = f[z(t)], \quad a \leq t \leq b$$

is the parametric representation of the curve $C^{\bullet}$ in the w-plane under the analytic mapping $w = f(z)$.

Let $z_1 = z(t_1)$ be a point in the nbd. of the point $z_0 = z(t_0)$ on C and $\Delta t = t_1 - t_0$. Now $\dot{z}(t_0)$ is the tangent vector to the curve C at the point t_0, given by

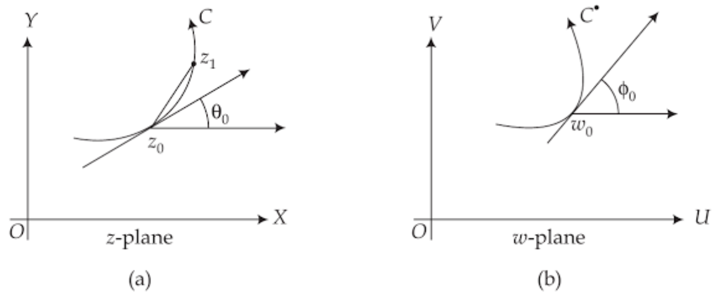


Fig. 14.13

$$\dot{z}(t_0) = \left.\frac{dz}{dt}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{z_1 - z_0}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{z(t_0 + \Delta t) - z(t_0)}{\Delta t}$$

The angle between this vector and the positive x-axis is $\arg(\dot{z}(t_0))$ and let $\theta_0 = \arg(\dot{z}(t_0))$.

Next, the point $z_0 = z(t_0)$ on C corresponds to the point $w(t_0) = w_0$ of $C^\bullet$ and $\dot{w}(t_0)$ represents a tangent vector to $C^\bullet$ at the point t_0. By chain rule

$$\frac{dw}{dt} = \frac{df}{dz} \cdot \frac{dz}{dt}$$

or, $$\dot{w}(t_0) = f'(z_0)\ \dot{z}(t_0) \qquad \text{...(14.51)}$$

Thus, if $f'(z_0) \neq 0$, then $\dot{w}(t_0) \neq 0$ and $C^\bullet$ has a unique tangent at $w(t_0)$, also the angle between the tangent vector $\dot{w}(t_0)$ and the positive u-axis is given by $\arg(\dot{w}(t_0))$. Since the argument of a product equals the sum of the arguments of the factors, thus from (14.51) we have

$$\arg\ \dot{w}(t_0) = \arg f'(z_0) + \arg\ \dot{z}(t_0)$$

or, $$\arg\ \dot{w}(t_0) - \arg\ \dot{z}(t_0) = \arg f'(z_0).$$

Thus, under the mapping $w = f(z)$ the tangent to C at z_0 is rotated through an angle $\arg f'(z_0)$, which is independent of the choice of C. That is, the transformation $w = f(z)$ rotates the tangents of all the curves through z_0 by the same angle $\arg f'(z_0)$. *Hence, the curves C_1 and C_2 through z_0 which form a certain angle at z_0 are mapped upon curves $C_1^\bullet$ and $C_2^\bullet$ forming the same angle in sense as well as in magnitude at the image point w_0 of z_0.*

This proves the angle-preserving property or conformality of mapping by analytic functions.

A mapping that preserves the magnitude of the angle but not necessarily the direction is called an 'isogonal mapping'.

Remarks 1. If $w = f(z) = u + iv$ defines a conformal mapping, then u and v must satisfy C–R equations and, therefore,

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{-\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2.$$

Hence, in a conformal transformation infinitesimal areas are magnified by the factor $J\left(\frac{u, v}{x, y}\right)$ and *also the condition of a conformal mapping is* $J\left(\frac{u, v}{x, y}\right) \neq 0.$

2. The practical importance of conformal mapping results from the fact that harmonic functions of two real variables remains harmonic under a change of variables arising a conformal transformation. This has important consequences. Suppose that it is required to solve a boundary value problem in connection with a two-dimensional potential, that is, to determine a solution of Laplace equation in two independent variables in a given region D which assumes given value on the boundary of D. It may be possible to find a conformal mapping which transforms D into some simpler region $D^\bullet$ such as a circular disk or a half-plane. Thus, we may solve Laplace's equation subject to the transformed boundary conditions in $D^\bullet$. The resulting solution when carried back to D by the use of that mapping will be the solution of the original problem.

14.7.2 Some Special Conformal Transformations

(a) Transformation: $w = z^2$

It is conformal everywhere except where $f'(z) = 2z = 0$.

We have $\quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$

It gives $\quad u = x^2 - y^2$ and $v = 2xy$

If $u = a$ is constant, then $x^2 - y^2 = a$. Similarly if $v = b$ is constant, then $xy = b/2$.

Both represent pair of rectangular hyperbolas. Hence, a pair of lines $u = a$, $v = b$ parallel to the axes in the w-plane map into pair of orthogonal rectangular hyperbolas in the z-plane as shown in Fig. 14.14a and 14.14b.

Similarly we can show that the pair of lines $x = c$ and $y = d$ parallel to the axes in the z-plane map into orthrogonal parabolas in the w-plane, as shown in Fig. 14.15a and 14.15b.

Also $\frac{dw}{dz} = 0$ gives $z = 0$, therefore, it is a critical point of the mapping.

Taking $z = re^{i\theta}$ and $w = \text{R}e^{i\phi}$, the transformation $w = z^2$ becomes $\text{R}e^{i\phi} = r^2e^{2i\theta}$, which gives

$$R = r^2 \text{ and } \phi = 2\theta.$$

Hence circles $r = r_0$ are mapped onto circles $R = r_0^2$ and rays $\theta = \theta_0$ onto rays $\phi = 2\theta_0$, as shown in Fig. 14.16a and 14.16b. In fact, the upper half of the z-plane $0 \leq \theta \leq \pi$ transforms into the entire w-plane $0 \leq \phi \leq 2\pi$. The same holds for the lower half. On the similar lines we can explain that the

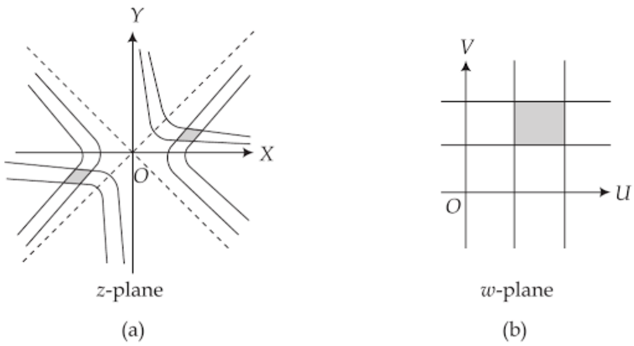


Fig. 14.14

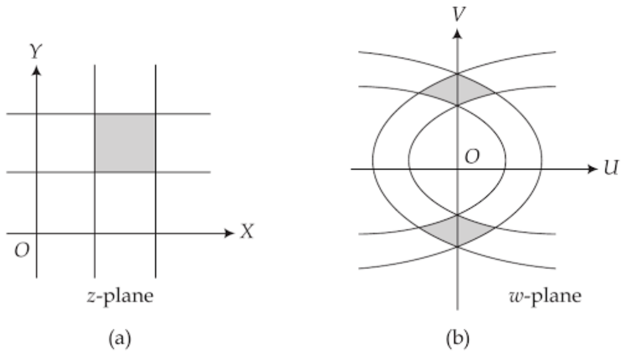


Fig. 14.15

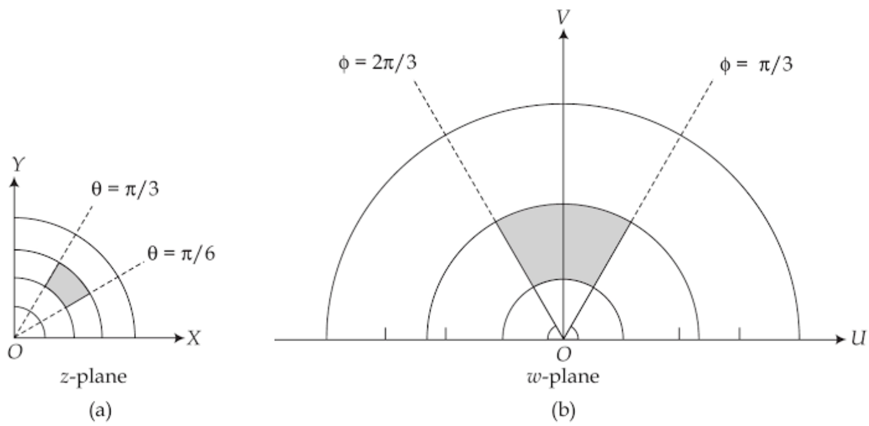


Fig. 14.16

transformation $w = z^n$, n being a positive integer is conformal mapping of the z-plane everywhere except at $z = 0$. The sector $0 \le \theta \le \pi/n$ is mapped by $w = z^n$ onto the upper half-plane $v \ge 0$.

(b) *Transformation:* $w = e^z$

Since $\dfrac{dw}{dz} = e^z \ne 0$ for any z, the mapping is conformal at every point z. Setting $z = x + iy$ and $w = \mathrm{Re}^{i\phi}$, we have $\mathrm{Re}^{i\phi} = e^{x+iy} = e^x e^{iy}$, which gives

$$R = e^x \text{ and } \phi = y.$$

Thus the lines parallel to y-axis, that is, x = constant in the z-plane maps into the circles R = constant in the w-plane, as shown in Fig. 14.17a and 14.17b. Their radii being less than or greater than 1 according as x is less than or greater than 0. Similarly the lines parallel to x-axis that is, y = constant in the z-plane maps into the radial lines ϕ = constant of the w-plane. For example, the rectangular region $ABCD$ given by $0 \le x \le 1$ and $0.5 \le y \le 1$ in the z-plane, as shown in Fig. 14.18a, is mapped onto the circular region $A'B'C'D'$ given by $1 \le R \le e$, and $0.5 \le \phi \le 1$ as shown in Fig. 14.18b.

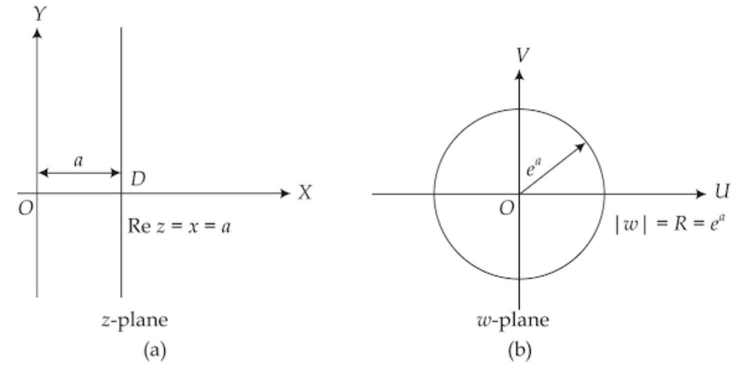


Fig. 14.17

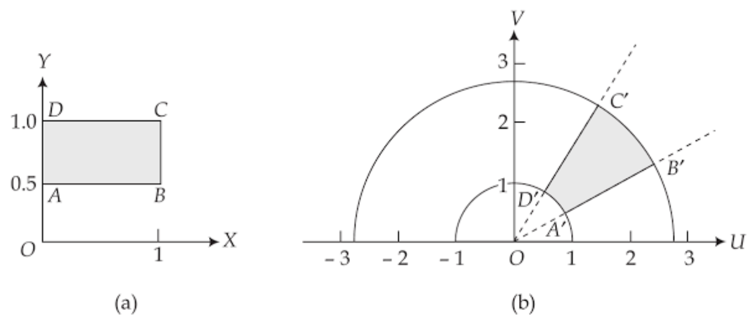


Fig. 14.18

Also we can easily verify that under $w = e^z$ any strip of height 2π in the z-plane will cover once the entire w-plane.

(c) *Joukowski transformation:* $w = z + 1/z$

In this case $\dfrac{dw}{dz} = 1 - \dfrac{1}{z^2}$, the mapping is conformal except at the points $1 - \dfrac{1}{z^2} = 0$, or $z = \pm 1$, which corresponds to the points $w = 2$ and $w = -2$ of the w-plane. Hence the points $z = \pm 1$ are the critical points. Changing to polar co-ordinates we have

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Therefore, $$u = \left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad v = \left(r - \frac{1}{r}\right)\sin\theta \qquad \text{...(14.52)}$$

Eliminating θ from these we obtain

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1 \qquad \text{...(14.53)}$$

From (14.53) it follows that family of circles r = constant in the z-plane maps onto the family of the ellipses in the w-plane. These ellipse are *confocal* since

$$\left(r + \frac{1}{r}\right)^2 + \left(r - \frac{1}{r}\right)^2 = 4,$$

which is independent of r, refer Figs. 14.19a and 14.19b.

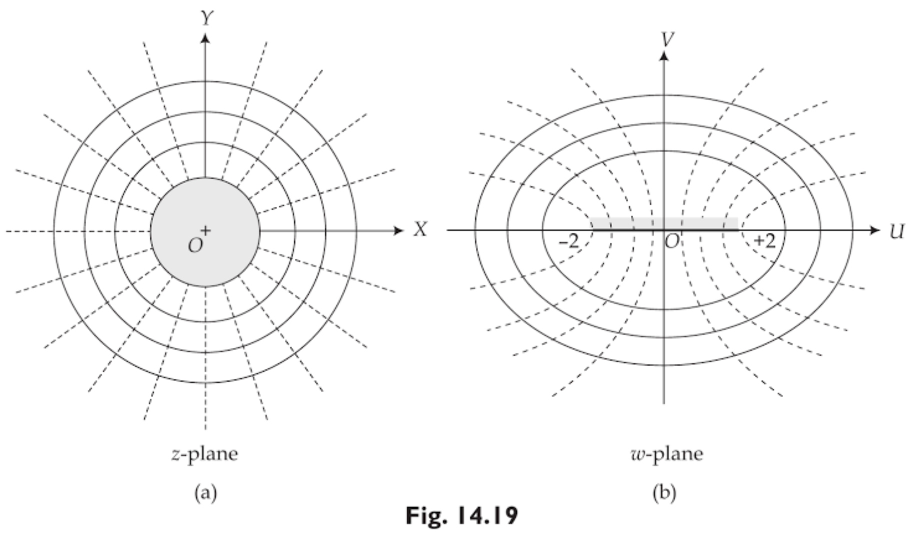


Fig. 14.19

In particular, the unit circle $r = 1$ in the z-plane maps into the segment $u = -2$ to $u = 2$ of the real axis in the w-plane.

Next eliminating r from (14.52) we have

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4$$

or,
$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad \text{...(14.54)}$$

From (14.54) it follows that the radial line θ = constant in the z-plane transforms into a family of hyperbolas which are also confocal. The transformations are shown in Figs. 14.19a and 14.19b.

(d) *Bilinear transformation*: $w = (az + b)/(cz + d)$, $ad - bc \neq 0$.
This is conformal everywhere except at $z = -d/c$ and $w = a/c$, since

$$\frac{dw}{dz} = \frac{(ad - bc)}{(cz + d)^2} \neq 0$$

because of the condition $ad - bc \neq 0$.

The points $z = -d/c$ and $w = a/c$, $c \neq 0$ are called the critical points of the transformation. Also it is conformal at $z = \infty$, since for $z = 1/\xi$, we get $w = \dfrac{b\xi + a}{d\xi + c}$ and it is easy to check that $\dfrac{dw}{d\xi} \neq 0$ as $\xi \to 0$.

(e) *Transformations:* $w = \sin z$, $\cos z$, $\sinh z$, $\cosh z$, $\tan z$ and $\tanh z$.

We have, $u + iv = \sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

Thus $u = \sin x \cosh y$ and $v = \cos x \sinh y$

Eliminating y from these, we obtain

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \qquad \text{...(14.55)}$$

Again eliminating x, we obtain

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \qquad \text{...(14.56)}$$

The Eqs. (14.55) and (14.56) show that the rectangular net of straight lines x = constant and y = constant is mapped onto a net of hyperbolas, (image of x = constant) and ellipses, (image of y = constant). Exceptions are the vertical lines $x = \pm\pi/2$ which are transformed onto $u \leq -1$ and $u \geq +1$, with $v = 0$, respectively. The mapping is shown in Figs. 14.20a and 14.20b.

Further, the upper semi-infinite strip $-\pi/2 \leq x \leq \pi/2$, $y \geq 0$ in the z-plane under $w = \sin z$ transforms to the upper half plane in the w-plan, refer Fig. 14.21a and 14.21b and the lower strip $-\pi/2 \leq x \leq \pi/2$, $y \leq 0$ maps to the lower half-plane of the w-plane.

The *mapping* $w = \cos z$ can be discussed on the similar lines as $\sin z$. Alternatively, we can view $\cos z$ as $\cos z = \sin(z + \pi/2)$. The mapping is a translation to the right through $\pi/2$ units given as $z_1 = z + \pi/2$ followed by the sine mapping $w = \sin z_1$.

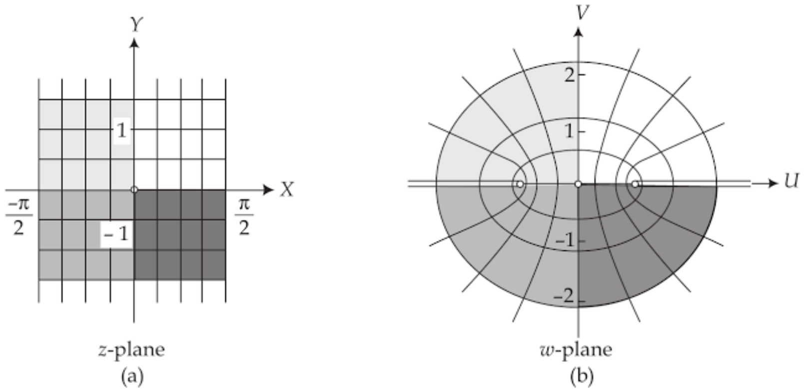


Fig. 14.20

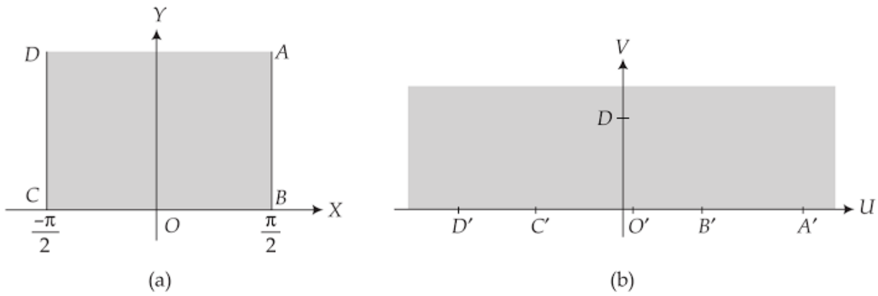


Fig. 14.21

The mapping $w = \sinh z$ can be expressed as $w = \sinh z = -i \sin(iz)$. This can be interpreted as an anticlockwise rotation $z_1 = iz$ through $\pi/2$, followed by the sine mapping $z_2 = \sin z$, and then followed by a clockwise rotation through $\pi/2$ given as $w = -iz_2$.

The *mapping $w = \cosh z$* $= \cos(iz)$ can be explained in terms of anticlockwise rotation through $\pi/2$ followed by the cosine mapping. However, to describe the transformation independently express it is

$$u + iv = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$$

which gives

$$u = \cosh x \cos y \quad \text{and} \quad v = \sinh x \sin y.$$

Eliminating x and y from these equations give respectively

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad \text{...(14.57)}$$

and

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1 \qquad \text{...(14.58)}$$

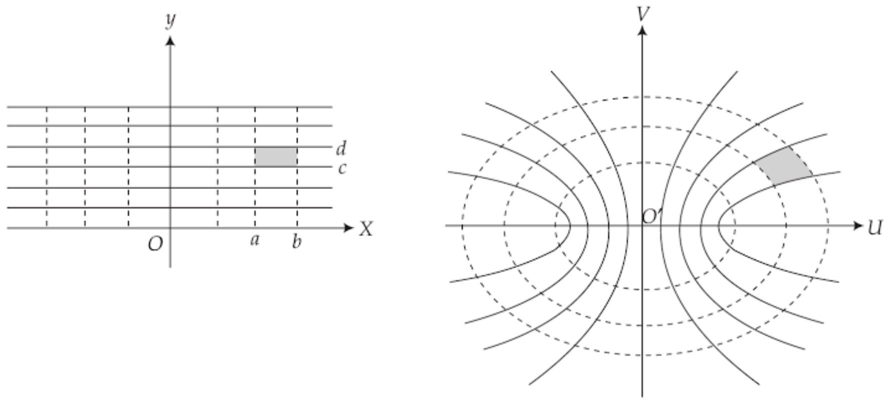


Fig. 14.22

Equation (14.57) shows that the line y = constant in the z-plane maps to a hyperbola in the w-plane, and the Eq. (14.58) shows that the line x = constant maps to ellipse in the w-plane. The rectangular region $a < x < b$, $a < y < d$ in the z-plane transforms into the shaded region in the w-plane as shown in Fig. 14.22a and 14.22b.

Next, the mapping $w = \tan z$ can be expressed as

$$w = \tan z = \frac{\sin z}{\cos z} = \frac{(e^{iz} - e^{-iz})/i}{e^{iz} + e^{-iz}} = \frac{-i(e^{2iz} - 1)}{e^{2iz} + 1} \quad \text{...(14.59)}$$

In case we set $z_1 = e^{2iz}$ and $z_2 = \dfrac{z_1 - 1}{z_1 + 1}$, then (14.59) becomes

$$w = \tan z = \frac{-i(z_1 - 1)}{(z_1 + 1)} = -iz_2$$

Thus the mapping $w = \tan z$ can be viewed as a bilinear transformation preceded by an exponential mapping and proceeded by a clockwise rotation through $\pi/2$.

Similarly, the mapping $w = \tanh z$ is expressed as

$$w = \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^z}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1} = \frac{z_1 - 1}{z_2 - 1},$$

where $z_1 = e^{2z}$ and thus can be explained accordingly.

Example 14.31: Determine the angle of rotation at the point $z = (1 + i)/2$ under the mapping $w = z^2$. Find its scale factor also.

Solution: The angle of rotation is given by $\psi_0 = \arg [f'(z_0)]$

Here $f(z) = z^2$ and $z_0 = (1+ i)/2$. Thus, $f'(z_0) = f'[\,|(1 + i)/2] = (1 + i)$.

Hence, $\psi_0 = \arg (1 + i) = \tan^{-1} (1) = \pi/4$.

Also the scale factor is, $a = |f'(z_0)| = |1 + i| = \sqrt{2}$

Example 14.32: Find the image of the infinite strip $0 \le x \le \pi/2$ in the z-plane under the mapping $w = \tan^2 z/2$.

Solution: We have $w = \tan^2(z/2) = \dfrac{\sin^2(z/2)}{\cos^2(z/2)} = \dfrac{1-\cos z}{1+\cos z}$

Also $\cos z = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

At $x = 0$, $\cos z = \cosh y$ and hence $w = \dfrac{1-\cosh y}{1+\cosh y}$ is purely real at $x = 0$ and thus

$$u = \frac{1-\cosh y}{1+\cosh y}.$$

At $y = 0$, $u = 0$ and for $0 < y < \infty$, u varies from 0 to –1. Also as $y \to -\infty$, u again tends to –1.

At $x = \pi/2$, $\cos z = -i \sinh y$ and hence $w = \dfrac{1+i\sinh y}{1-i\sinh y}$. It gives

$$|w| = \frac{|1+i\sinh y|}{|1-i\sinh y|} = \frac{\sqrt{1+\sinh^2 y}}{\sqrt{1-\sinh^2 y}} = 1, \text{ for all } y.$$

Thus the line $x = \pi/2$ in the z-plane is mapped into the unit circle $|w| = 1$ in the w-plane.

For any line $x = a$, when $0 < a < \pi/2$, we have

$$|w| = \left|\frac{1-\cos(a+iy)}{1+\cos(a+iy)}\right| = \frac{\sqrt{(1-\cos a\cosh y)^2 + (\sin a \sinh y)^2}}{\sqrt{(1+\cos a\cosh y)^2 + (\sin a \sinh y)^2}} < 1, \text{ for all } y.$$

Thus as x goes from 0 to a, $0 < a < \pi/2$, the interior of the circle $|w| = 1$ is mapped. The mapping is shown in Fig. 14.23a and 14.23b.

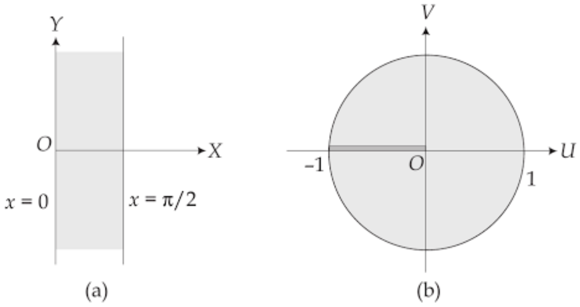


Fig. 14.23

14.8 SCHWARZ-CHRISTOFFEL TRANSFORMATION

The Schwarz-Christoffel transformation maps the interior of a polygon P which can be a triangle, rectangle, or other polygon, bounded or unbounded into the upper-half of the z-plane and conversely. The boundary of the polygon is mapped into the real axis.

Let the polygon have vertices $w_1, w_2, \ldots, w_n$ in the w-plane and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be its interior angles as shown in Fig. 14.24a. Then the Schwarz-Christoffel transformation is given by

$$w = a\int (z - x_1)^{\frac{\alpha_1}{\pi}-1} (z - x_2)^{\frac{\alpha_2}{\pi}-1} \ldots (z - x_n)^{\frac{\alpha_n}{\pi}-1} dz + b, \qquad \ldots(14.60)$$

where $x_1, x_2, \ldots, x_n$ are the points on the real axis corresponding to the vertices $w_1, w_2, \ldots, w_n$ of the polygon P; a and b are constants and integral in (14.60) is taken over any path from z_0 to z in the upper half-plane.

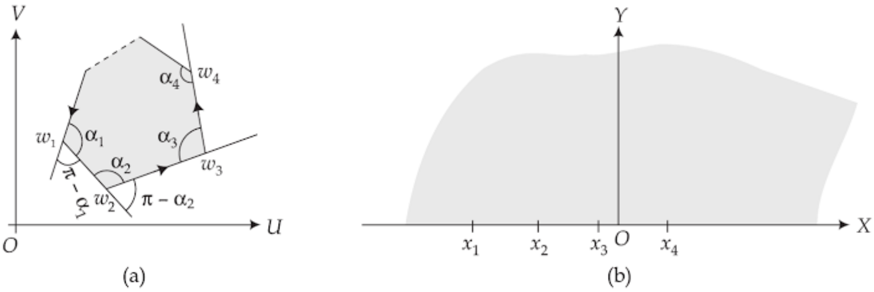


Fig. 14.24

From (14.60), we have

$$\frac{dw}{dz} = a(z - x_1)^{\frac{\alpha_1}{\pi}-1} (z - x_2)^{\frac{\alpha_2}{\pi}-1} \ldots (z - x_n)^{\frac{\alpha_n}{\pi}-1} \qquad \ldots(14.61)$$

Thus,

$$\arg\left(\frac{dw}{dz}\right) = \arg(a) + \left(\frac{\alpha_1}{\pi} - 1\right)\arg(z - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right)\arg(z - x_2) + \ldots + \left(\frac{\alpha_n}{\pi} - 1\right)\arg(z - x_n) \qquad \ldots(14.62)$$

Now imagine z moving from left to right along the real axis, refer Fig. 14.24b. When z moves from $-\infty$ to x_1, suppose that w moves along the side $w_n w_1$ of the polygon P towards w_1 and hence no change in the angle. As z passes over x_1 from left to right then in (14.62), the arg $(z - x_1)$ changes from π to 0 but all other terms remain unaffected. Hence the arg $\frac{dw}{dz}$ decreases by $\left(\frac{\alpha_1}{\pi} - 1\right)\pi = \alpha_1 - \pi$, or increases by $\pi - \alpha_1$ in the anticlockwise direction. Thus the direction at w_1 turns by the angle $(\pi - \alpha_1)$ in the positive direction along w_1w_2. Next this angle remains unchanged as z moves from x_1 towards x_2 but as it passes over x_2 the argument increases by $\pi - \alpha_2$ in the anticlockwise direction along w_2w_3 and so on. Thus, we observe that as z moves along x-axis, w traces the polygon $w_1w_2 \ldots w_n$ and the real axis is mapped to a polygon with exterior angles $\pi - \alpha_1, \pi - \alpha_2, \ldots, \pi - \alpha_n$, where

$$\sum_{j=1}^{n} (\pi - \alpha_j) = 2\pi, \quad \text{or} \quad \sum_{j=1}^{n} \alpha_j = (n - 2)\pi.$$

In the case of an unbounded polygon, the vertex w_n may be considered as a point at ∞. Then the transformation (14.60) is modified to

$$w = a\int (z - x_1)^{\frac{\alpha_1}{\pi}-1} (z - x_2)^{\frac{\alpha_2}{\pi}-1} \ldots (z - x_{n-1})^{\frac{\alpha_{n-1}}{\pi}-1} dz + b \qquad ...(14.63)$$

Example 14.33: Find the transformation which maps the semi-infinite strip of breadth d in the w-plane ($u \geq 0$) into the upper half of the z-plane.

Solution: The semi-infinite strip $ABCD$ as shown in Fig. 14.25a can be considered as the limiting case of a triangle with two vertices at B and C and the third vertex A (or D) at infinity

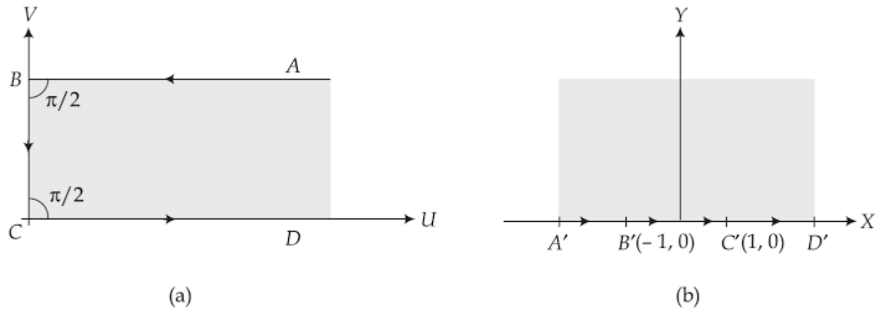


Fig. 14.25

Let the vertices B and C map into the points $B'(-1, 0)$ and $C'(1, 0)$ of the z-plane as shown in Fig. 14.25b. The interior angles at B and C are $\pi/2$, hence the Schwarz-Christoffel transformation becomes

$$w = a\int (z+1)^{\frac{\pi/2}{\pi}-1} (z-1)^{\frac{\pi/2}{\pi}-1} dz + b$$

$$= a\int \frac{dz}{\sqrt{z^2 - 1}} + b = a\cosh^{-1} z + b \qquad ...(14.64)$$

When $z = 1$, then $w = 0$. Thus $0 = a\cosh^{-1}(1) + b$, which gives $b = 0$, since $\cosh^{-1}(1) = 0$

When $z = -1$, then $w = id$. Thus $id = a\cosh^{-1}(-1)$, or, $\cosh(id/a) = -1$, or, $\cos(d/a) = \cos\pi$, or $a = d/\pi$. Substituting for a and b in (14.64), the requisite transformation is

$$w = \frac{d}{\pi}\cosh^{-1} z, \quad \text{or} \quad z = \cosh\frac{\pi w}{d}$$

Example 14.34: Find the transformation which maps the strip given by Im $(w) \geq 0$, $-c < \text{Re}(w) < c$ in the w-plane into the upper half of the z-plane.

Solution: The strip $\text{Im}(w) \geq 0$, $-c < \text{Re}(w) < c$ is shown in Fig. 14.26a. This can be considered a polygon with vertices at $A(-c, 0)$, $B(c, 0)$ and ∞.

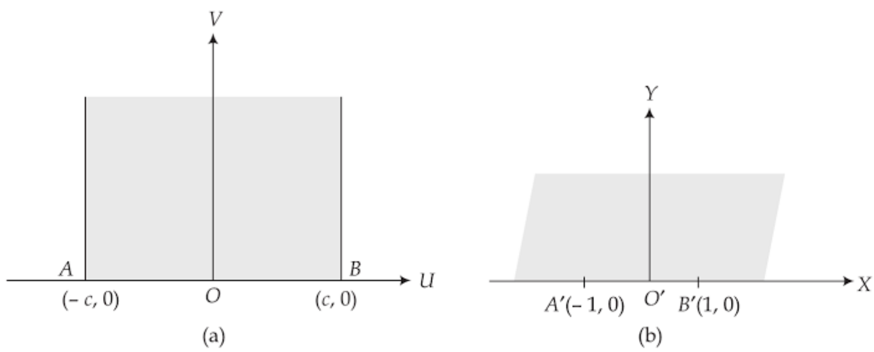


Fig. 14.26

Let the vertices A and B map into the points $A'(-1, 0)$ and $B'(1, 0)$ of the z-plane as shown in Fig. 14.26b. The interior angles at A and B are $\pi/2$. The Schwarz-Christoffel transformation becomes

$$w = a\int (z+1)^{\frac{\pi/2}{\pi}-1}(z-1)^{\frac{\pi/2}{\pi}-1}\,dz + b$$

$$= -ia\int \frac{1}{\sqrt{1-z^2}}\,dz + b = -ia\sin^{-1} z + b \qquad \text{...(14.65)}$$

When $z = -1$, then $w = -c$. Thus (14.65) gives, $-c = -ia \sin^{-1}(-1) + b$, or $-c = ia\,\pi/2 + b$

Similarly, at $z = 1$, $w = c$ and we obtain, $c = -ia\,\pi/2 + b$

These give $b = 0$ and $-ia = \dfrac{2c}{\pi}$. Substituting in (14.65), the requisite transformations is

$$w = \frac{2c}{\pi}\sin^{-1}(z), \text{ or } z = \sin\frac{\pi w}{2c}.$$

EXERCISE 14.5

1. Show that the bilinear transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ is conformal at $z = \infty$.
2. Discuss the transformation $w = \sqrt{z}$. Is it conformal at the origin?
3. Under the transformation $w = 1/z$, show that the image of the hyperbola $x^2 - y^2 = 1$ is a lemniscate.
4. (a) Show that transformation $w = z + (1/z)$ converts the straight line $\arg z = \alpha$, $|\alpha| < \pi/2$ into a branch of hyperbola of eccentricity $\sec\alpha$.
   (b) Show that $w = (z + 1/z)/2$ maps $|z| = r$ to an ellipse if $0 < r \leq 1$.
5. Show that transformation $w = \sin z$ maps the upper semi-infinite strip $-\pi/2 \leq x \leq \pi/2$, $y \geq 0$ to the upper half-plane in the w-plane and the lower semi-infinite strip $-\pi/2 \leq x \leq \pi/2$, $y \leq 0$ to the lower half-plane.

6. Find and sketch the image of $2 \le |z| \le 3$, $\pi/4 \le \theta \le \pi/2$ under the mapping $w = Ln(z)$.

7. Discuss the transformation $w = e^z$ and show that it transforms the region $0 \le y \le \pi$ into the upper-half of the w-plane.

8. Determine the points where the following mappings are not conformal

(a) $\bar{z}$ (b) $\cos z$ (c) $\sin \pi z$ (d) $\cosh z$

9. The interior of a square with vertices at 0, 1, $1 + i$ and i in the z-plane is mapped onto a region R under the mapping $w = z + 2 - 3i$. Show that the mapping is conformal and the interior angles of the mapped region R are at right angle.

10. Find the images of the circle $(x - 3)^2 + (y - 2)^2 = 2$ and the line $x + 2y = 8$ under the transformation $w = 1/z$. Show that the images of the circle and the line intersect at the same angle in the w-plane as in the z-plane.

11. Show that under the transformation $w = \cos z$, the infinite strip given by $c \le x \le d$ when $c, d \in (0, \pi/2)$ is mapped into the region between the two branches of the hyperbola lying in the right half of the w-plane.

12. Show that under the conformal mapping $w = u + iv = f(z)$, $f(z)$ being analytic, any harmonic function $\phi(x, y)$ is transformed into another harmonic function.

13. Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0$, $v = \pi$ and $u = 0$ onto the upper-half of the z-plane.

14. Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0$, $v = \pi$ onto the upper-half of the plane.

15. Show that the Schwarz-Christoffel transformation

$$f(z) = 2i \int_0^z (z+1)^{-1/2} (z-1)^{-1/2} z^{-1/2}\, dz$$

maps the upper-half-plane onto the rectangle with vertices o, c, $c + ic$ and ic, where $c = \Gamma(1/2)\,\Gamma(1/4)\,\Gamma(3/4)$. Here Γ is the gamma function.

ANSWERS

Exercise 14.1 (p. 10)

1. (a) Closed disk, center $-2 - 5i$, radius 1/2, connected, not a domain, bounded.
   (b) $y \ge 0$, connected, unbounded, not open, not a domain.
   (c) $x \le y < \infty$, connected, unbounded, not open, not a domain,
   (d) Between the branches of the hyperbola, connected, unbounded, not open, not a domain
   (e) Horizontal infinite strip of width 2π, connected, unbounded, open, defines a domain.
   (f) $|z| < 1/2$, open disk, center (0, 0), radius 1/2, connected, bounded, not a domain.
2. (a) Complex plane, except $z = i, -1$
   (b) Complex plane, except $z = \cos(k\pi/2) + i \sin(k\pi/2)$, $k = 0, 1, 2, 3$
   (c) Complex plane, except the circle $|z| = 1$

3. (a) $2 < u < 3, 1 < v < 2.$ (b) $\frac{u^2}{4} - 1 < v < 1 - \frac{u^2}{4}, -2 < u < 0.$

4. (a) $-3/20, 1/20$ (b) $-1/5, 7/5$ (c) $1, 8.$

6. (a) $-4i$ (b) $-i/2$ (c) does not exist (d) does not exist

7. No.

8. (a) discontinuous (b) discontinuous (c) continuous

10. (a) $-1/(z+1)^2$ (b) $-2/z^3$ (c) not differentiable (d) $2/(1-z)^2$

Exercise 14.2 (p. 22)

2. (a) $\frac{1}{2}(e^{-1} + e)\cos 2,$ (b) $e^{-2\pi n}$ (c) $e^{i \ln \sqrt{2} - (\pi/4 + 2n\pi)}$ (d) $e^{\pi/2 + 2n\pi}$

3. (a) $z = -i[\ln|1 \pm \sqrt{2}| + i\{(\pm\pi/2) + 2n\pi\}]$ (b) $z = [2n\pi - \tan^{-1}(4/3)]/2$
   (c) $z = [\ln(1/3) + i(2n+1)\pi/2]$ (d) $z = \ln|\alpha| + i[2n\pi + \text{Arg}(\alpha)]$

4. (a) $(\pi + 2\pi n)i,$ (b) $\pi/2 + 2\pi k - i\ln(2 \pm \sqrt{3})$ (c) $2\pi n$

7. $\sqrt{(\cos 2x + \cosh 2y)/2} - i\tan^{-1}(\tan x \tanh y).$

Exercise 14.3 (p. 38)

1. (a) Analytic (b) Not analytic (c) Not analytic
   (d) Not analytic (e) Not analytic (f) Not analytic

4. (a) $1/z + ic$ (b) $e^{-x}(\cos y - i\sin y) + c$ (c) $ze^{2z} + ic$
   (d) $z \sin z$ (e) $\bar{z}e^{-\bar{z}+c}$ (f) $\cot z + ic$

5. (a) $f(z) = (1 - \cot(z/2))/2$ (b) ize^{-z}

6. (a) $f(z) = z^2 + c$ (b) $f(z) = z + (1/z) + ic$

Exercise 14.4 (p. 46)

1. A triangle with vertices $(4, -1)$, $(7, 1)$ and $(7, -3)$
2. A rectangle with vertices $(0, -2)$, $(1, -3)$, $(3, -1)$ and $(2, 0)$
3. A straight line $1 + 4v = 0$ in the w-plane.
5. $w = (1 + iz)/(1 - iz)$ invariant points are $z = -(1 + i \pm \sqrt{6}i)/2$
7. (a) $w = i(1-z)/(1+z);$ (b) $w = (iz - 2)/(z + 2)$ (c) $w = (1-z)/(1+z)$
8. Image is $u \le 1/2$ and $v \ge 0.$
9. (i) $w = (az + b)/(bz + a)$, a, b, arbitrary
   (ii) $w = (az - c)/(cz + a)$, a, c arbitrary

10. If z_0 is the upper-half of the z-plane show that the bilinear transformation $w = e^{i\alpha}(z - z_0)/(z - \bar{z}_0)$ maps the upper-half of the z-plane into the interior of the unit circle at the origin in the w-plane.

Exercise 14.5 (p. 59)

6. $\ln 2 \le u \le \ln 3, \pi/4 \le v \le \pi/2$.
8. (a) nowhere (b) $z = n\pi, n = 0, \pm 1, \ldots$
   (c) $z = (2n + 1)/2, n = 0, \pm 1, \ldots$ (d) $z = n\pi i, n = 0, \pm 1, \ldots$
10. $11(u^2 + v^2) - 6u + 4v + 1 = 0$ and $8(u^2 + v^2) - u + 2v = 0$; the common angle of intersection is $\tan^{-1}\sqrt{3}$.
13. $z = \cosh w$ 14. $w = Ln\,(z)$

15 Complex Integration

CHAPTER

"The concept of definite integral of real functions does not directly extend to the case of complex functions, since real functions are usually integrated over intervals and complex functions are integrated over curves. Surprisingly, complex integrations are not so complex to evaluate, oftenly simpler than the evaluation of real integrations. Some real integrals which are otherwise difficult to evaluate can be evaluated easily by complex integration, and moreover, some basic properties of analytic functions are established by complex integration only."

15.1 LINE INTEGRAL IN THE COMPLEX PLANE

The concept of definite integral $\int_a^b f(x)dx$, as studied in calculus of a real valued function f on a real variable x, was generalized to line integral as applied to vector field in Chapter 9(Vol I). Here we extend the concept once more and consider the line integral of a complex function. As in calculus of a real variable, here also we distinguish between definite integrals and indefinite integrals. Complex definite integrals are called the *line integrals* and are written as

$$\int_C f(z)dz$$

The integrand $f(z)$ is integrated over a given curve C in the complex plane called the *path of integration* normally represented by a parametric representation,

$$z(t) = x(t) + iy(t), \qquad a \le t \le b.$$

The sense of increasing t is called the positive sense on C. The curve C is assumed to be *smooth* curve, that is, it has continuous and non-zero derivative at each $t \in (a, b)$. In case the initial point and terminal point of a curve coincide, that is $z(a) = z(b)$, the curve is said to be closed one.

15.1.1 The Complex Line Integral

The definition of the complex line integral is similar to that of definite integral in calculus of a real variable.

Consider a smooth curve C in the complex plane given by $z(t) = x(t) + iy(t)$, $a \le t \le b$ and $f(z)$ a continuous function defined at each point of C. Divide C into n parts at the points $z_0, z_1, \ldots, z_{n-1}, z_n$ corresponding to the partition $t_0(=a), t_1, \ldots, t_{n-1}, t_n(=b)$, such that $t_0 < t_1 < \ldots < t_n$ of the interval $a \le t \le b$, as shown in Fig. 15.1.

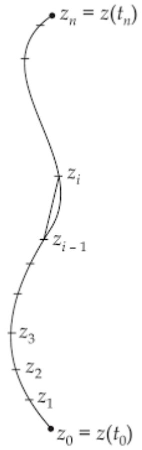


Fig. 15.1

Let $\Delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $z_{i-1}\, z_i$ between z_{i-1} and z_i Consider the sum

$$S_n = \sum_{i=1}^{n} f(\xi_i)\Delta z_i$$

The limit of the sum S_n as $n \to \infty$ in such a way that the length of the chord Δz_i approaches zero is called the line integral of $f(z)$ taken along the curve C oriented from z_0 to z_n and is denoted by

$$I = \int_C f(z)dz \qquad \ldots(15.1)$$

In case the path of integral C is a closed curve, then the integration is denoted by $\oint_C f(z)dz$ and the integral along a closed curve is sometimes called the *contour integral*.

In general, the path of integration for complex line integrals are assumed to be *piecewise smooth*, that is, consisting of finitely many smooth curves joined end to end.

Next, writing $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + idy$, the line integral (15.1) can be expressed as

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy) \qquad \ldots(15.2)$$

Thus the line integral of a complex function can be evaluated in terms of two line integrals of real functions.

15.1.2 Basic Properties of Line Integrals

A few basic properties of the line integrals of a complex function which follow directly from the definition are given below:

1. *Linearity:* $\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1\int_C f_1(z)dz + k_2\int_C f_2(z)dz,$

where k_1 and k_2 are two constants.

2. ***Sense reversal:*** $\int_{z_o}^{z} f(z)dz = -\int_{z}^{z_o} f(z)dz$

3. ***Partitioning of path:*** $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz,$

where the curve C consists of two smooth curves C_1 and C_2 joined end to end as shown in Fig. 15.2.

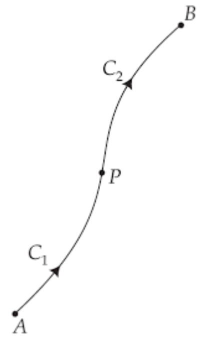


Fig. 15.2

4. ***ML-inequality:*** $\left|\int_C f(z)dz\right| \le ML,$

where M is a constant such that $|f(z)| \le M$ everywhere on C and L is the length of the curve.

Example 15.1: Evaluate $\int_C z^2 dz$, where C is the straight line joining the origin O to the point $P(2, 1)$ in the complex plane.

Solution: The equation of the line OP is $x = 2y$, $0 \le y \le 1$.

Thus, $\quad dz = dx + idy = 2dy + idy = (2 + i)dy.$

Also, $\quad z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = 3y^2 + 4iy^2.$

Hence,
$$\int_C z^2 dz = \int_0^1 (3 + 4i)y^2(2 + i)dy = (2 + 11i)\int_0^1 y^2 dy = \frac{1}{3}(2 + 11i).$$

Example 15.2: Evaluate $\oint_C (z - a)^n dz$, where 'a' is a given complex number, n is any integer and C is a circle of radius R centered at 'a' and oriented anticlockwise.

Solution: It is convenient here to use parametric equation of the circle in the form
$$C: z - a = \text{Re}^{i\theta},\ 0 \le \theta \le 2\pi, \text{ so } dz = i\text{Re}^{i\theta}\, d\theta.$$

Thus,
$$\oint_C (z - a)^n dz = \int_0^{2\pi} R^n e^{ni\theta}\, iRe^{i\theta}\, d\theta = iR^{n+1}\int_0^{2\pi} e^{(n+1)i\theta}\, d\theta$$
$$= R^{n+1}\left|\frac{e^{(n+1)i\theta}}{n+1}\right|_0^{2\pi} = \frac{R^{n+1}}{n+1}\left[e^{2(n+1)i\pi} - 1\right] = 0, \text{ provided } n \ne -1.$$

For $n = -1$, we have $\oint_C \frac{dz}{z - a} = \int_0^{2\pi} \frac{1}{\text{Re}^{i\theta}}\, i\text{Re}^{i\theta}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$

Example 15.3: Evaluate the integral $\int_0^{1+i} (x - y + ix^2)dz$

(a) along the straight line from $z = 0$ to $z = 1 + i$

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$.

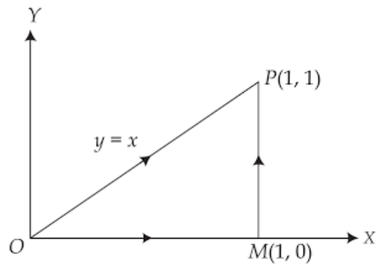


Fig. 15.3

Solution: (a) The equation of the straight line OP, refer Fig. 15.3, is $y = x$. Thus along the line OP, $z = x + iy = x + ix = (1 + i)x$, which gives $dz = (1 + i)dx$, $0 \le x \le 1$, and hence

$$\int_0^{1+i} (x - y + ix^2)dz = \int_0^1 (x - x + ix^2)(1 + i)dx = i(1 + i)\int_0^1 x^2 dx = -\frac{1}{3}(1 - i).$$

(b) Along the path OM, we have $y = 0$ and thus $z = x + iy = x$ and hence $dz = dx$, $0 \le x \le 1$. Also, along the path MP, we have $x = 1$ and thus $z = x + iy = 1 + iy$, and hence $dz = idy$, $0 \le y \le 1$.

Therefore, the line integral

$$\int_0^{1+i} (x - y + ix^2)dz = \int_0^1 (x + ix^2)dx + \int_0^1 (1 - y + i)\,(idy).$$

$$= \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1 + \left[(i-1)y - \frac{iy^2}{2}\right]_0^1 = \frac{1}{2} + \frac{i}{3} + (i - 1) - \frac{i}{2} = -\frac{1}{2} + \frac{5}{6}i.$$

Example 15.4: Evaluate $\oint_C \ln z\, dz$, where C is the unit circle $|z| = 1$ taken in counter clockwise sense.

Solution: Any point on the unit circle $|z| = 1$ in parametric form is $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, which gives $dz = ie^{i\theta}d\theta$. Thus the line integral becomes

$$\oint_C \ln zdz = \int_0^{2\pi} \ln e^{i\theta}.ie^{i\theta}d\theta = -\int_0^{2\pi} \theta e^{i\theta}d\theta = -\left[\theta\frac{e^{i\theta}}{i} - 1.\frac{e^{i\theta}}{i^2}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{2\pi i}}{i} + e^{2\pi i} - 1\right] = -\frac{2\pi}{i} = 2\pi i.$$

Example 15.5: Evaluate $\oint_C |z|^2 dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1).

Solution: The contour of integration C is $OABCO$ as shown in Fig. 15.4.

We have, $|z|^2 = (x^2 + y^2)$, and also along

$$\begin{array}{lllll} OA: & y=0, & 0 \le x \le 1, & dz = dx, & |z|^2 = x^2 \\ AB: & x=1, & 0 \le y \le 1, & dz = i\,dy, & |z|^2 = 1+y^2 \\ BC: & y=1, & x \text{ goes from 1 to 0}, & dz = dx, & |z|^2 = 1+y^2 \\ CO: & x=0, & y \text{ goes from 1 to 0}, & dz = i\,dy, & |z|^2 = y^2 \end{array}$$

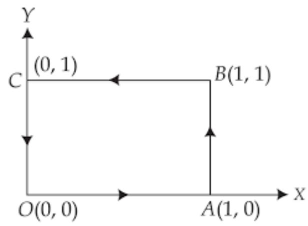


Fig. 15.4

Thus,
$$\oint_C |z|^2 dz = \int_0^1 x^2 dx + i\int_0^1 (1+y^2)dy + \int_1^0 (1+x^2)dx + i\int_1^0 y^2 dy$$
$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = -1 + i.$$

Example 15.6: Find an upper bound to the integral $I = \int_C \frac{e^z}{z^2}dz$, where C is the straight line from (0, 1) to (2, 0) in the complex plane.

Solution: The path C is the line segment AB as shown in Fig. 15.5. Consider

$$|f(z)| = \left|\frac{e^z}{z^2}\right| = \frac{|e^{x+iy}|}{|x+iy|^2} = \frac{|e^x||e^{iy}|}{x^2+y^2} = \frac{e^x}{x^2+y^2} \quad \text{...(15.3)}$$

On C, e^x is maximum at $x = 2$, so maximum value of e^x is e^2.

Next the minimum value of $x^2 + y^2$ on C is the square of OP, the perpendicular distance from O to the line AB given by $x + 2y - 2 = 0$. This is $\left(2/\sqrt{5}\right)^2 = 4/5$.

Thus from (15.3) we have, $|f(z)| \le \frac{5e^2}{4}$. Also L, the length $|AB| = \sqrt{5}$.

Using the ML-inequality, we have

$$\left|\int_C \frac{e^z}{z^2}dz\right| \le \frac{5e^2}{4}(\sqrt{5}) = 20.65$$

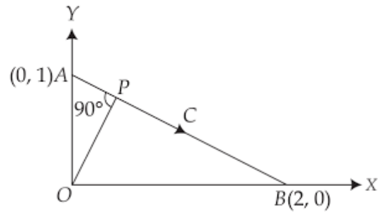


Fig. 15.5

EXERCISE 15.1

1. Evaluate $\int_C z^2 dz$, where C is the curve given by

(a) $z(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ 2 + i(t-1), & 1 \le t \le 2 \end{cases}$

(b) the straight line joining the point (1, 1) to the point (2, 4) on the complex plane.
(c) the parabola $x = 4 - y^2$ as y goes from 2 to -2.

2. Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along, (a) the line $y = x/2$,

(b) the real axis to 2 and then vertically to $2 + i$.

3. Evaluate $\int_C (z - z^2)dz$, where C is the upper half of the circle $|z - 2| = 3$.

4. Evaluate $\int_C z \operatorname{Re}(z)dz$, where C: $z(t) = t - it^2$ for $0 \le t \le 2$.

5. Evaluate $\int_C \sin^2 z\, dz$, where C: $|z| = \pi$, $\operatorname{Re} z \ge 0$, from $-\pi i$ to πi in the right half-plane.

6. Evaluate $\int_C \cos z\, dz$, where C is the semicircle $|z| = \pi$, $x \ge 0$ from $-\pi i$ to πi.

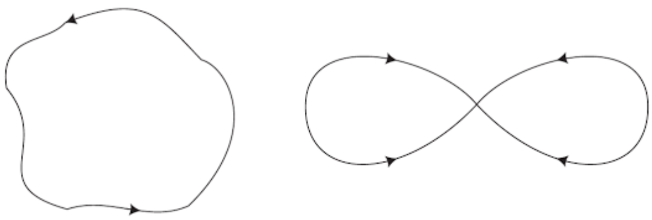
7. Find an upper bound to the integral $\int_C \frac{\sin z}{z(z^2+9)} dz$, where C: $|z| = 5$.

8. If C is a straignt line from $z = 2i$ to $z = 3$, show that $\left|\int_C \frac{\cos z}{z} dz\right| \le \frac{13}{6} \cosh 2$.

15.2 CAUCHY'S INTEGRAL THEOREM. INDEPENDENCE OF PATH

Cauchy's integral theorem is considered the fundamental theorem of complex integration because of its important consequences. This theorem extends the idea of contour integration further. Before introducing this theorem we need to introduce a few concepts concerning the types of domains.

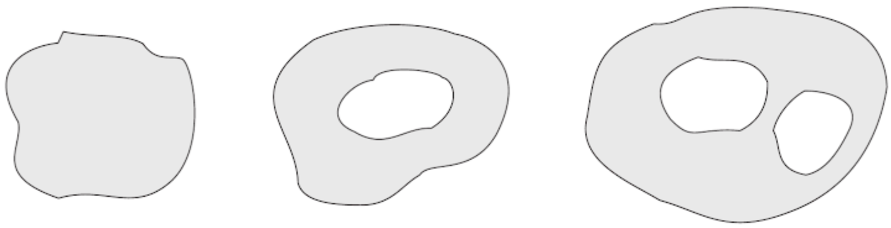
Simple closed path. A *simple closed path* is a closed path that does not intersect or touch itself. For example, an ellipse, or a circle are simple closed paths. But the self-intersecting an 8-shaped curve is not a simple closed path, refer Fig. 15.6a and 15.6b.



(a) (b)

Fig. 15.6

Simply connected domain. A *simply connected domain D* in the complex plane is a domain such that every simple closed path in D encloses only points of D. A domain that is not simply connected is called *multiply connected*. For example, interior of an ellipse, or of a circle are examples of simply connected domains while interior of an annulus, for example $1 < |z| < 2$, is boubly connected domain. Figures 15.7a, 15.7b and 15.7c represent respectively simply, doubly and triply connected domains.



(a) (b) (c)

Fig. 15.7

Now we are in a position to state the Cauchy's integral theorem.

Theorem 15.1: (Cauchy's integral theorem) *If f(z) is analytic and f′(z) is continuous in a simply connected domain D, then for every piecewise smooth closed curve C in D the contour integral*

$$\oint_C f(z)dz = 0 \qquad \ldots(15.4)$$

Proof. Writing $f(z) = u + iv$ and $dz = dx + idy$, we have

$$\oint_C f(z)dz = \oint_C (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy) \qquad \ldots(15.5)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in D, and hence in the region enclosed by C. Thus Green's theorem, refer Section 9.4, is applicable to the right side of (15.5) and hence it becomes

$$\oint_C f(z)dz = -\iint_E \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\iint_E \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy, \qquad \ldots(15.6)$$

where E is the region bounded by the closed curve C, refer Fig. 15.7a.

Since $f(z)$ is analytic, u and v satisfy the Cauchy-Riemann equations (14.22), and thus the integrands of the two double integrals on the right side of (15.6) are identically zero and hence we obtain (15.4).

We must note that *analyticity of f(z) is only sufficient but not a necessary condition for (15.4) to be true.*

We can check very easily that $\oint_C \frac{dz}{z^2} = 0$, where C is the unit circle, refer Example (15.2) for $a = 0$ and $n = -2$, but this result does not follow from Cauchy's theorem since $f(z) = 1/z^2$ is not analytic in $|z| < 1$, zero being the point of singularity.

On the other hand, *simple connectedness of the domain is essential one*. For example, $\oint_C \frac{dz}{z} = 2\pi i$, where C is the unit circle lying in the annulus $1/2 < |z| < 3/2$, refer Example (15.2). Here, $f(z) = 1/z$ is analytic in the given domain but this domain is not simply connected so Cauchy theorem is not applicable.

Example 15.7: Evaluate the following integrals by applying Cauchy's integral theorem, in case applicable

(a) $\oint_C \cos z\, dz$ (b) $\oint_C \sec z\, dz$ (c) $\oint_C \frac{dz}{z^2 - 5z + 6} dz$ (d) $\oint_C \bar{z}\, dz$,

where C is the unit circle $|z| = 1$.

Solution: (a) The integrand $f(z) = \cos z$ is analytic for all z and also $f'(z) = \sin z$ is continuous everywhere, and hence on and inside C also. Thus by Cauchy's theorem

$$\oint_C \cos z\, dz = 0.$$

(b) The integrand $f(z) = \sec z = \frac{1}{\cos z}$ is not analytic at the points $z = \pm \pi/2, \pm 3\pi/2, \ldots$ but all these points lie outside the unit circle $|z| = 1$. Hence $f(z)$ is analytic and $f'(z)$ is continuous in and on C, and thus

$$\oint_C \sec z\, dz = 0.$$

(c) The integrand $f(z) = \frac{1}{z^2 - 5z + 6} = \frac{1}{(z-2)(z-3)}$ is analytic everywhere except at $z = 2, 3$, the points which lie outside the unit circle $|z| = 1$, and hence by Cauchy's theorem

$$\oint_C \frac{1}{z^2 - 5z + 6} dz = 0.$$

(d) The integrand $f(z) = \bar{z}$ is not analytic and hence the Cauchy's theorem is not applicable. In fact, about C: $|z| = 1$, we have

$$\oint_C \bar{z}\, dz = \int_0^{2\pi} e^{-i\theta}\, ie^{i\theta}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

15.2.1 Independence of Path

In the preceding section, we have noted that a line integral of a function $f(z)$ depends not merely on the end points of the path but also the path itself, refer Example (15.3). We say that an integral of $f(z)$ is *independent of path in a domain D*, if for every z_1, z_2 in D the value of $\int_{z_1}^{z_2} f(z)dz$ depends only on the end points z_1 and z_2 and not on the choice of the path C joining z_1 to z_2. An important consequence of Cauchy's theorem is to look for the situations when the line integral is independent of path in a domain D. We have the following result:

Theorem 15.2: (Independence of path) *If $f(z)$ is analytic in a simply connected domain D, then $\int_C f(z)dz$ is independent of the path for every piecewise smooth curve C lying entirely within D.*

Proof. Let $P(z_1)$ and $Q(z_2)$ be any two points in D and let C_1 and C_2 be two arbitrary paths in D from P to Q intersecting each other only at the end points P and Q, as shown in Fig. 15.8a. Consider the curve $C_2^{\bullet}$ same as C_2 but with reverse orientation as shown in Fig. 15.8b. We observe that $C_1 \cup C_2^{\bullet}$ is a piecewise smooth simple closed curve in D, and so according to Cauchy's integral theorem

$$\int_{C_1 \cup C_2^{\bullet}} f(z)dz = 0, \text{ which gives } \int_{C_1} f(z)dz = -\int_{C_2^{\bullet}} f(z)dz, \text{ or } \int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

The minus sign disappears in case we integrate in the reverse direction.

This proves the theorem.

In case the two paths have finitely many points in common as shown in Fig. 15.9, then the independence of path can be proved by applying the argument to each loop separately.

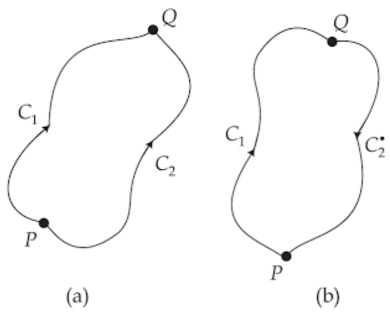


Fig. 15.8

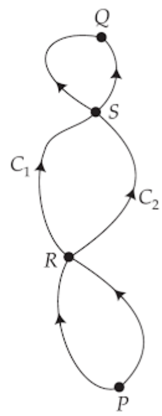


Fig. 15.9

15.2.2 Deformation of Path

It is useful to consider path independence in terms of process of *path deformation*. We can visualize deforming C_1 continuously into C_2, refer Fig. 15.10, keeping the end points P and Q fixed. If f is

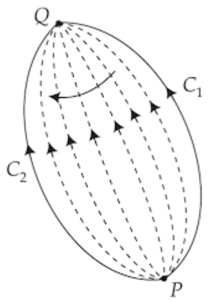
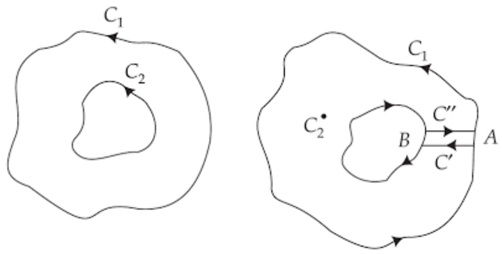


Fig. 15.10



(a) (b)

Fig. 15.11

analytic on C_1 and C_2 and we cross no singular point in the process of deformation from C_1 to C_2, then the line integral $\int_{C_1} f(z)dz$ is equal to $\int_{C_2} f(z)dz$, and the latter may be more easily evaluated than the former.

The path deformation can be applied to closed paths as well and this gives the *extension of Cauchy's theorem to doubly connected regions*.

Consider $f(z)$ to be analytic on and between the two closed paths C_1 and C_2 as shown in Fig. 15.11a. Slit the region enclosed by C_1 and C_2 by introducing a piecewise smooth curve connecting a point A on C_1 with a point B on C_2. Denote the curve A to B as C' and from B to A as C'' as shown in Fig. 15.11b. Consider the curve C as

$$C = C_1 + C' + C_2^\bullet + C''$$

The function $f(z)$ is analytic inside and on C, thus from Cauchy's theorem $\oint_C f(z)dz = 0$, which gives

$$\oint_{C_1} f(z)dz + \int_{C'} f(z)dz + \oint_{C_2^\bullet} f(z)dz + \int_{C''} f(z)dz = 0$$

or,
$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

since,
$$\oint_{C_2^\bullet} f(z)dz = -\oint_{C_2} f(z)dz \quad \text{and} \quad \int_{C''} f(z)dz = -\int_{C'} f(z)dz$$

Thus, we have the following theorem

Theorem 15.3 (Extension of the Cauchy's integral theorem): *If $f(z)$ is analytic on and between two closed paths C_1 and C_2, then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \qquad \ldots(15.7)$$

This result can be extended to multiply connected regions also as shown in Fig. 15.12. The result is as follow:

Theorem 15.4: *If $f(z)$ is analytic on and between the region included in the closed curves C, C_1, C_2, C_3 etc., then*

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \ldots . \qquad \ldots(15.8)$$

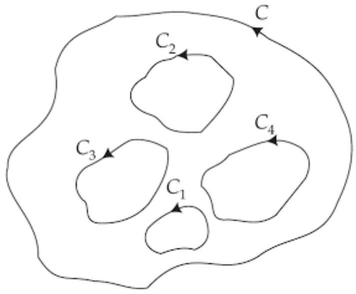


Fig. 15.12

Example 15.8: Verify that the line integral $I = \int_C z^2 dz$ is the same in each of the following cases

(a) C is the straight line OP joining the points $O(0, 0)$ and $P(1, 2)$.
(b) C is the straight line from $O(0, 0)$ to $A(1, 0)$ and then from $A(1, 0)$ to $P(1, 2)$.
(c) C is the parabolic path $y = 2x^2$.

Solution: The three paths are shown in the Fig. 15.13.

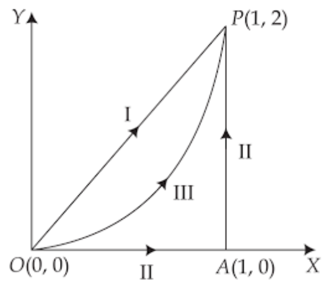


Fig. 15.13

(a) Equation of the line OP is $y = 2x, 0 \le x \le 1$.

Thus, $z^2 = (x + iy)^2 = (1 + 2i)^2 x^2$ and $dz = dx + idy = (1 + 2i)dx$

Therefore, $I = \int_C z^2 dz = \int_0^1 (1 + 2i)^3 x^2\, dx = \frac{1}{3}(1 + 2i)^3 = -\frac{1}{3}(11 + 2i)$

(b) Along OA; $y = 0, 0 \le x \le 1, z = x$. Thus we have $z^2 = (x + iy)^2 = x^2$ and $dz = dx$.

Along AP; $x = 1, 0 \le y \le 2$, thus we have $z = 1 + iy$, $z^2 = (1 + iy)^2$ and $dz = idy$.

Therefore, $$I = \int_0^1 x^2 dx + i\int_0^2 (1 + iy)^2 dy = \frac{1}{3} + \frac{(1+2i)^3}{3} - \frac{1}{3} = -\frac{1}{3}(11 + 2i).$$

(c) Along the curve $y = 2x^2, 0 \le x \le 1$, we have, $z = x + iy = x + 2ix^2$, thus $z^2 = (1 + 2ix)^2 x^2$ and $dz = dx + 4ix\, dx = (1 + 4ix)dx$

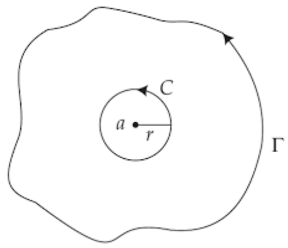
Therefore, $$I = \int_C z^2 dz = \int_0^1 (1 + 2ix)^2 x^2 (1 + 4ix)dx = \int_0^1 (x^2 - 4x^4 + 4ix^3)(1 + 4ix)dx$$

$$= \int_0^1 [(x^2 - 20x^4) + i(8x^3 - 16x^5)]dx = -\frac{1}{3}(11 + 2i)$$

Thus along all the three paths the value of the line integral is the same. In fact the integrand z^2 is analytic in the entire complex plane, the value of the line integral I depends only on the end points.

Example 15.9: Evaluate $\oint_{\Gamma} \frac{1}{z-a}\, dz$ over any closed path enclosing the given point 'a'.

Solution: Figure 15.14 shows a typical such path but it cannot be parameterized, since we do not know the contour Γ specifically. Let C be a circle of radius r with centre 'a'. Since the function $f(z)$ is analytic on and between Γ and C, thus by principle of deformation of path



$$\int_{\Gamma} \frac{1}{z-a}\, dz = \int_{C} \frac{1}{z-a}\, dz .$$

The contour C can be parametrized as $z = a + re^{i\theta}$, $0 \le \theta \le 2\pi$, and thus

Fig. 15.14

$$\int_{\Gamma} \frac{1}{z-a}\, dz = \int_{C} \frac{1}{z-a}\, dz = \int_{0}^{2\pi} \frac{e^{-i\theta}}{r} . ir\, e^{i\theta}\, d\theta = i \int_{0}^{2\pi} d\theta = 2\pi i.$$

In general, we note that for any closed anticlockwise contour Γ about a point 'a', we have

$$\oint_{\Gamma} (z-a)^n\, dz = \begin{cases} 2\pi i\,, & n = -1 \\ 0\,, & n \neq -1 \end{cases} \qquad \ldots(15.9)$$

This result follows from the principle of deformation and Example (15.2).

Example 15.10: Evaluate $I = \oint_{C} \frac{dz}{z^2(z-2)(z-4)}$, where C is the rectangle joining the points $(-1, -1)$, $(3, -1)$, $(3, 1)$ and $(-1, 1)$ in the complex plane.

Solution: The curve C is the rectangle $ABCD$ as shown in Fig. 15.15. Expanding the integrand in the partial fractions, we obtain

$$I = \frac{3}{32}\oint_{C} \frac{dz}{z} + \frac{1}{8}\oint_{C} \frac{dz}{z^2} - \frac{1}{8}\oint_{C} \frac{dz}{z-2} + \frac{1}{32}\oint_{C} \frac{dz}{z-4} \qquad \ldots(15.10)$$

$$= \frac{3}{32}(2\pi i) + \frac{1}{8}(0) - \frac{1}{8}(2\pi i) + \frac{1}{32}(0) = -\frac{\pi i}{16}$$

The first three integrals on the right side of (15.10) are evaluated by using (15.9), and the last integral is zero by Cauchy's integral theorem.

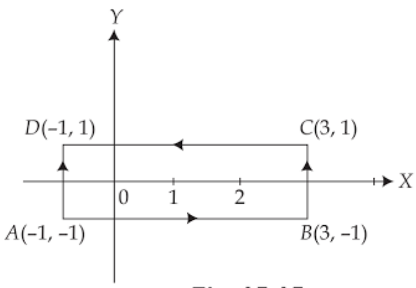


Fig. 15.15

Example 15.11: Evaluate the integral $\oint_{C} \frac{dz}{z(z+2)}$, where C is any rectangle containing the points $z = 0$ and $z = -2$ inside it.

Solution: The integrand $f(z)$ is analytic everywhere except at the points $z = 0$ and $z = -2$, both the points lying inside the rectangle C. Draw circles C_1 and C_2 respectively enclosing the points $z = 0$ and

$z = -2$ as shown in Fig. 15.16. The function $f(z)$ is analytic on and between the curves C, C_1 and C_2 and hence by the extension of Cauchy's theorem we have

$$\oint_C \frac{dz}{z(z+2)} = \oint_{C_1} \frac{dz}{z(z+2)} + \oint_{C_2} \frac{dz}{z(z+2)} = \frac{1}{2}\left[\oint_{C_1} \frac{dz}{z} - \oint_{C_1} \frac{dz}{z+2} + \oint_{C_2} \frac{dz}{z} - \oint_{C_2} \frac{dz}{z+2}\right] \quad \ldots(15.11)$$

By Cauchy's theorem, $\oint_{C_1} \frac{dz}{z+2} = 0, \int_{C_2} \frac{dz}{z} = 0.$

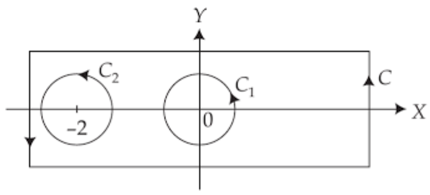


Fig. 15.16

Also, $\int_{C_1} \frac{dz}{z} = 2\pi i$ and $\int_{C_2} \frac{dz}{z+2} = 2\pi i,$

refer Example (15.9). Therefore, from (15.11), we obtain

$$\oint_C \frac{dz}{z(z+2)} = \frac{1}{2}(2\pi i - 2\pi i) = 0$$

15.3 EXISTENCE OF INDEFINITE INTEGRAL: FUNDAMENTAL THEOREM OF THE COMPLEX INTEGRAL CALCULUS

In this section, we discuss the existence of the indefinite integral of a function $f(z)$ and give the *fundamental theorem of the complex integral calculus*, a result analogous to the fundamental theorem of integral calculus. The theorem is useful for evaluating the integrals for which an antiderivative can be found simply by inspection. The theorem is stated as follows.

Theorem 15.5 (Fundamental theorem of complex integral calculus): *If $f(z)$ is analytic in a simply connected domain D and z_0 be any fixed point in D, then*

$$F(z) = \int_{z_0}^{z} f(z^\bullet)dz^\bullet$$

is analytic in D given by $F'(z) = f(z)$, and

$$\int_{z_0}^{z} f(z^\bullet)dz^\bullet = F(z) - F(z_0). \quad \ldots(15.12)$$

Proof. Since the conditions of Cauchy's integral theorem are satisfied hence the line integral of $f(z)$ from point z_0 in D to any point z in D is independent of path in D. Keeping z_0 fixed, this integral becomes a function of z, say $F(z)$. Thus,

$$F(z) = \int_{z_0}^{z} f(z^\bullet)dz^\bullet,$$

which is uniquely determined.

Next we show that $F(z)$ is analytic in D and $F'(z) = f(z)$. For this consider

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^\bullet)dz^\bullet - \int_{z_0}^{z} f(z^\bullet)dz^\bullet\right]$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^\bullet)dz^\bullet = \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z)+f(z^\bullet)-f(z)]dz^\bullet$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)dz^\bullet + \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z^\bullet)-f(z)]dz^\bullet$$

$$= f(z) + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^\bullet)-f(z)]dz^\bullet$$

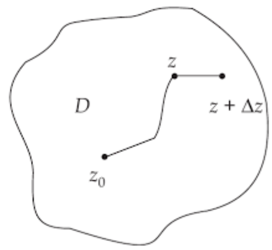


Fig. 15.17

or, $$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z^\bullet)-f(z)]dz^\bullet \qquad \ldots(15.13)$$

The points z_0, z, $z + \Delta z$ are shown in Fig. 15.17. Because of path independence principle, without loss of generality, the path from z to $z + \Delta z$ may be taken as straight line.

Using the ML-inequality on the right side of (15.13), we obtain

$$\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z^\bullet)-f(z)]dz^\bullet\right| \le \frac{1}{\Delta z} M\,|\,\Delta z\,| = M,$$

where $M = \max.\,|f(z^\bullet) - f(z)|$ on the straight line from z to $z + \Delta z$. Since $f(z)$ is analytic and therefore, continuous also, thus $M \to 0$ as $\Delta z \to 0$, and, hence letting $\Delta z \to 0$ in (15.13) gives

$$F'(z) = f(z) \qquad \ldots(15.14)$$

A function F(z) satisfying F′(z) = f(z) is called an 'indefinite integral' or 'primitive' of f.

Next, we show that any two primitives of a given function f differ at most by an arbitrary additive constant. For this, let $G(z)$ be an other primitive of $f(z)$, therefore, $G'(z) = f(z)$, and then

$$G'(z) - F'(z) = f(z) - f(z) = 0, \text{ and hence } G(z) - F(z) = c \text{ in } D,$$

where c is an arbitrary constant.

Thus, in general, for a specific primitive $G(z)$ of f, we have

$$G(z) = \int_{z_0}^{z} f(z^\bullet)\, dz^\bullet = F(z) + c.$$

To evaluate c put $z = z_0$, we obtain $0 = F(z_0) + c$ which gives, $c = -F(z_0)$,

and so

$$\int_{z_0}^{z} f(z^{\bullet})\,dz^{\bullet} = F(z) - F(z_0)$$

This completes the proof.

Example 15.12: Evaluate the following integrals

$$\text{(a)} \int_{2i}^{3} \sin z\,dz \qquad \text{(b)} \int_{0}^{1} z^2 e^{z^3} dz \qquad \text{(c)} \int_{0}^{2i} \sinh z\,dz$$

Solution: The functions $\sin z$, $z^2 e^{z^3}$ and $\sinh z$ are analytic everywhere, therefore, the integrals can be evaluated by applying indefinite integration.

(a) $$F(z) = \int_{2i}^{3} \sin z\,dz = \left[-\cos z\right]_{2i}^{3} = -\cos 3 + \cos 2i = \cosh 2 - \cos 3.$$

(b) $$F(z) = \int_{0}^{1} z^2 e^{z^3} dz = \frac{1}{3}\int_{0}^{1} e^t dt = \frac{1}{3}\left[e^t\right]_0^1 = \frac{1}{3}(e-1)$$

(c) $$F(z) = \int_{0}^{2i} \sinh z\,dz = \left[\cosh z\right]_0^{2i} = \cosh 2i - 1 = \cos 2 - 1.$$

EXERCISE 15.2

1. Verify Cauchy's integral theorem for the integral of z^2 over the boundary of the square with vertices $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$ taken counterclockwise.
2. Can the Cauchy's integral theorem be applied for evaluating the following integrals? If so, evaluate, if not, evaluate otherwise

   (a) $\oint_C e^{\sin z^2} dz$; C: $|z| = 1$ (b) $\oint_C \frac{e^z}{z^2+9} dz$; C: $|z| = 2$

   (c) $\oint_C \frac{3z+5}{z(z+2)} dz$; C: $|z| = 1$ (d) $\oint_C \frac{dz}{z^2}$; C: $|z| = \frac{1}{2}$

3. Evaluate $\oint_\Gamma \left(\frac{4}{z-1} - \frac{5}{z+4}\right) dz$, where Γ is any square of side 3 units with its centre at the origin.
4. Evaluate the following integrals around the unit circle C: $|z| = 1$ indicating whether Cauchy's integral theorem applies

(a) $\oint_C e^{-z^2}\, dz$ (b) $\oint_C \frac{dz}{|z|^2}$ (c) $\oint_C \text{Im}\, z\, dz$

5. Evaluate the following integrals, C taken in counterclockwise sense

(a) $\oint_C \frac{e^z}{z^2 - 5iz - 6} dz$; C: $|z| = 1$ (b) $\oint_C \left(z + \frac{3}{z^2}\right) dz$; C: $|z| = 1$

(c) $\oint_C \frac{\cosh^2 2z}{(z + 3i)(z^2 + 16)} dz$; C: $|z| = 2$

6. Evaluate the following integrals using the extension of the Cauchy's integral theorem to multiply connected domains

(a) $\oint_C \frac{2z - 3}{z^2 - 3z - 18} dz$; C: $|z| = 8$ (b) $\oint_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2} dz$; C: $|z - 2| = 4$

(c) $\oint_C \frac{dz}{(z - 1)(z - 2)(z - 3)}$; C: $|z| = 4$

7. By evaluating $\oint_C e^z dz$, C: $|z| = 1$, show that $\int_0^{2\pi} e^{\cos\theta} \cos(\theta + \sin\theta) d\theta = 0$ and $\int_0^{2\pi} e^{\cos\theta} \sin(\theta + \sin\theta) d\theta = 0$.

8. Prove that $\int_C (z^2 + 2)^2 dz = 8\pi a(12\pi^4 a^4 + 20\pi^2 a^2 + 15)/15$, where C is the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ joining the points $(0, 0)$ and $(2\pi a, 0)$.

9. Show that the integral $\int_C e^{-2z}\, dz$, where C is the path joining the points $z = 1 + 2\pi i$ and $z = 3 + 4\pi i$ is independent of the path of integration. Evaluate it by taking a suitable path.

10. Use the fundamental theorem to evaluate the following integrals:

(a) $\int_i^0 \cos 3z\, dz$ (b) $\int_0^{3i} z e^{z^2}\, dz$ (c) $\int_0^{1+2i} z \sin(z^2) dz$

(d) $\int_0^{1+\pi i} (z^2 + \cosh 2z) dz$ (e) $\int_0^1 \frac{\tan^{-1} z}{1 + z^2} dz$ (f) $\int_{-i}^{i} z \cosh^2 z\, dz$

15.4 CAUCHY'S INTEGRAL FORMULA. DERIVATIVES OF AN ANALYTIC FUNCTION

Cauchy's integral formula is an important consequence of Cauchy's integral theorem. This gives a representation of an analytic function $f(z)$ at any interior point z_0 of a simply connected domain D as a contour integral evaluated along the boundary of a simple closed curve C which lies inside D and encloses the point z_0. The result is of fundamental importance and is stated as follows.

Theorem 15.6 (Cauchy's integral formula): *Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz, \qquad \ldots(15.15)$$

the integration being taken counter-clockwise.

Proof. Let C_1 be a circle with centre z_0 and radius r lying entirely within C. The function $\dfrac{f(z)}{z - z_0}$ is analytic on and within the closed curves C and C_1 as shown in Fig. 15.18, thus by the extension of Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{z - z_0}dz = \oint_{C_1} \frac{f(z)}{z - z_0}dz = \oint_{C_1} \frac{[f(z_0) + f(z) - f(z_0)]}{z - z_0}dz$$

$$= f(z_0)\oint_{C_1} \frac{dz}{z - z_0} + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}dz \qquad \ldots(15.16)$$

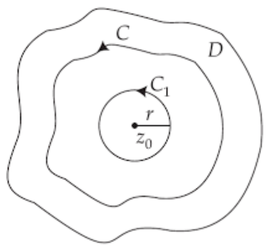


Fig. 15.18

Consider the first integral on the right side of (15.16). Put $z - z_0 = r\,e^{i\theta}$, we have $dz = ir\,e^{i\theta}\,d\theta$, and hence

$$\oint_{C_1} \frac{dz}{z - z_0} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

Next, if I denotes the second integral on the right side of (15.16), then

$$|I| = \left|\oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}dz\right| \le \oint_{C_1} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |dz| = \oint_{C_1} \frac{|f(z) - f(z_0)|}{|z - z_0|}|dz| \qquad \ldots(15.17)$$

Since $f(z)$ is continuous in D, (for it is analytic in D), thus for a given $\in > 0$, there exists a number $\delta > 0$ such that $|f(z) - f(z_0)| < \in$, wherever $|z - z_0| < \delta$.

Choosing the radius r of the circle C_1 such that $r < \delta$ and hence from (15.17), we have

$$|I| \le \oint_{C_1} \frac{|f(z) - f(z_0)|}{|z - z_0|}|dz| < \oint_{C_1} \frac{\in}{r}|dz| = \frac{\in}{r}2\pi r = 2\pi\in$$

Since $\in > 0$ can be chosen arbitrary small, thus $|I|$ can be made arbitrary small tending to zero, and thus Eq. (15.16) becomes

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0), \text{ or } f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz,$$

which is (15.15).

Example 15.13: Evaluate the integral $\oint_C \frac{z^2 + 1}{z^2 - 1} dz$, C: $|z - 1| = 1$

Solution: Writing the integrand as $\frac{z^2 + 1}{z^2 - 1} = \frac{(z^2 + 1)/(z + 1)}{z - 1}$

We observe that $f(z) = (z^2 + 1)/(z + 1)$ is analytic on and inside C, and here $z_0 = 1$, as shown in Fig. 15.19. Hence by Cauchy's integral formula

$$\oint_C \frac{z^2 + 1}{z^2 - 1} dz = 2\pi i f(1) = 2\pi i.$$

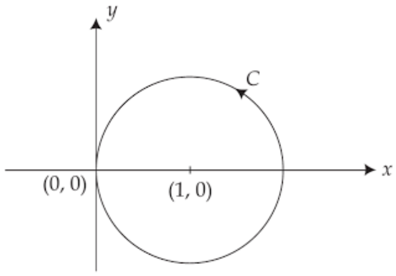


Fig. 15.19

Example 15.14: Evaluate the integral $\oint_C \frac{z^2 + 1}{z(2z - 1)} dz$, C: $|z| = 1$

Solution: Let $I = \oint_C \frac{z^2 + 1}{z(2z - 1)} dz$.

The integrand $(z^2 + 1)/z(2z - 1)$ is not analytic at the point $z = 0$ and $z = 1/2$ both of which lie inside C. Writing it as

$$\frac{z^2 + 1}{z(2z - 1)} = (z^2 + 1)\left[\frac{1}{(z - 1/2)} - \frac{1}{z}\right]$$

Therefore, $I = \oint_C \frac{z^2 + 1}{z - 1/2} dz - \oint_C \frac{z^2 + 1}{z} dz = 2\pi i\, [z^2 + 1]_{z = ½} - 2\pi i\, [z^2 + 1]_{z = 0} = \frac{5\pi i}{2} - 2\pi i = \frac{\pi i}{2}$,

using the Cauchy's integral formula.

Example 15.15: Evaluate the integral $\oint_C \frac{dz}{(z - z_0)(z - z_1)}$, where the points z_0 and z_1 lie inside the simple closed curve C and integration is taken in counter-clockwise sense.

Solution: Let C_0 and C_1 be two small simple closed non-intersecting curves surrounding z_0 and z_1 respectively and lying entirely within C. Then by the extension of the Cauchy's integral theorem, we have

$$\oint_C \frac{dz}{(z-z_0)(z-z_1)} = \oint_{C_0} \frac{dz}{(z-z_0)(z-z_1)} + \oint_{C_1} \frac{dz}{(z-z_0)(z-z_1)} \qquad \ldots(15.18)$$

Consider the first integral on the right side of (15.18), we have

$$\oint_{C_0} \frac{dz}{(z-z_0)(z-z_1)} = \oint_{C_0} \frac{dz/(z-z_1)}{z-z_0} = \frac{2\pi i}{(z_0-z_1)}, \text{ using Cauchy's formula.}$$

Similarly, $\oint_{C_1} \frac{dz}{(z-z_0)(z-z_1)} = \frac{2\pi i}{z_1 - z_0}$. Hence (15.18) becomes

$$\oint_C \frac{dz}{(z-z_0)(z-z_1)} = \frac{2\pi i}{z_0-z_1} + \frac{2\pi i}{z_1-z_0} = 0.$$

Example 15.16: Evaluate the integral $\oint_C \frac{\tan z}{z^2-1} dz$, C: $|z| = 3/2$, where integration is taken counter-clockwise.

Solution: The function $\tan z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \ldots$ but all these points lie outside the curve C: $|z| = 3/2$. Further $(z^2 - 1)^{-1}$ is not analytic at $z = 1$ and $z = -1$, both of these lie inside C. Writing the integrand as

$$\frac{\tan z}{z^2-1} = \frac{1}{2}\left[\frac{\tan z}{z-1} - \frac{\tan z}{z+1}\right]$$

and thus we have,

$$I = \oint_C \frac{\tan z}{z^2-1} dz = \frac{1}{2}\left[\oint_C \frac{\tan z}{z-1} dz - \oint_C \frac{\tan z}{z+1} dz\right] = \frac{2\pi i}{2}[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= 2\pi i \tan^{-1}(1) = \frac{\pi^2 i}{2}.$$

15.4.1 Derivatives of an Analytic Function

We now apply Cauchy's integral formula to show that *if a complex function f(z) is analytic in a domain D, then its derivatives of all orders exist and are also analytic in D*. This result is an important departure when compared to the real functions. A real function that is differentiable need not have a second derivative, and if it has a second, it need not have a third, and so on. In case of the derivatives of analytic functions we have the following result.

Theorem 15.7 (Generalized Cauchy's integral formula): *If f(z) is analytic in a domain D, then it has derivatives of all orders in D which are also analytic in D and the values of these derivatives at a point z_0 in D are given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz,\ n = 1, 2, ..., \qquad \text{...(15.19)}$$

where C is any simple closed path in D taken in counter-clockwise sense.

Proof. The Cauchy's integral formula is

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz.$$

Differentiating it under the integral sign w.r.t. z_0 we obtain

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz \qquad \text{...(15.20)}$$

Similarly, $$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz \text{ and, in general,}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

This completes the proof.

Remark. *Since we are not having a complex variable version of the Leibnitz rule for differentiation under the integral sign, the step at (15.20) lacks some justification. However, a rigorous proof of the result for the same can be given as follow:*

By the definition of differentiability of f(z) at $z = z_0$, we have

$$f'(z_0) = \lim_{\Delta z_0 \to 0} \frac{f(z_0+\Delta z_0) - f(z_0)}{\Delta z_0} \qquad \text{...(15.21)}$$

By Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \text{ and } f(z_0+\Delta z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-(z_0+\Delta z_0)}dz$$

Hence, $$\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i \Delta z_0}\oint_C \left[\frac{1}{z-(z_0+\Delta z_0)} - \frac{1}{z-z_0}\right]f(z)dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)\,[z-(z_0+\Delta z_0)]} \qquad \text{...(15.22)}$$

Consider

$$\frac{1}{(z-z_0)(z-z_0-\Delta z_0)} = \frac{1}{(z-z_0)^2}\left[\frac{z-z_0-\Delta z_0+\Delta z_0}{z-z_0-\Delta z_0}\right] = \frac{1}{(z-z_0)^2}\left[1+\frac{\Delta z_0}{z-z_0-\Delta z_0}\right]$$

$$= \frac{1}{(z-z_0)^2} + \frac{\Delta z_0}{(z-z_0)^2\,(z-z_0-\Delta z_0)} \qquad \text{...(15.23)}$$

Therefore, from (15.21), (15.22) and (15.23), we have

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz + \frac{1}{2\pi i}\lim_{\Delta z_0 \to 0}\oint_C \frac{\Delta z_0 f(z)}{(z-z_0)^2\,(z-z_0-\Delta z_0)}dz \qquad \ldots(15.24)$$

Since f(z) is analytic and hence continuous on C, it is bounded on C and thus $|f(z)| \le M$ *for all z on C. Let d be the shortest distance of* z_0 *from any point on C. Then*

$$|z-z_0| \ge d, \text{ which gives } \frac{1}{|z-z_0|^2} \le \frac{1}{d^2} \text{ and thus}$$

$$|z-z_0-\Delta z_0| \ge |z-z_0| - |\Delta z_0| \ge d - |\Delta z_0|, \text{ which gives } \frac{1}{|z-z_0-\Delta z_0|} \le \frac{1}{d-|\Delta z_0|}.$$

If L be the length of the curve C, then using the ML-inequality we obtain

$$\left|\frac{1}{2\pi i}\oint_C \frac{\Delta z_0 f(z)}{(z-z_0)^2\,(z-z_0-\Delta z_0)}\right| \le \frac{ML\,|\Delta z_0|}{2\pi d^2(d-|\Delta z_0|)} \to 0, \text{ as } \Delta z_0 \to 0,$$

and hence $f'(z_0)$ *exists and we obtain from (15.24) that*

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz \qquad \ldots(15.25)$$

Since z_0 *is any point in D, hence* $f'(z)$ *is analytic in D, and* $f'(z_0)$ *is given by (15.25). Repeating this argument we can, in general, prove that derivatives of all order exist which themselves are analytic and their values at a point* z_0 *in D are given by (15.19).*

Example 15.17: Evaluate the integral $\oint_C \frac{e^z}{z^3}dz$, C: $|z| = 1$, taken in counter-clockwise sense.

Solution: Let $I = \oint_C \frac{e^z}{z^3}dz$. Here $f(z) = e^z$ is analytic in the region bounded by the simple closed curve $|z| = 1$. The singular point $z = 0$ of $1/z^3$ lies inside $|z| = 1$. Hence, applying the generalized Cauchy's integral formula

$$I = \oint_C \frac{e^z}{z^3}dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}(e^z)\Big|_{z=0} = \pi i$$

Example 15.18: Evaluate $\oint_C \frac{(z+1)}{z(z-2)\,(z-4)^3}dz$, C: $|z-3| = 2$ in the counter- clockwise sense.

Solution: Let $I = \oint_C \frac{(z+1)}{z(z-2)\,(z-4)^3}dz$.

The integrand has singularities at $z = 0, 2$, and 4, out of these $z = 2$ and 4 lie inside C.

Consider two non-intersecting closed contour C_1 and C_2, as shown in Fig. 15.20, lying completely within C, respectively about the point $z = 2$ and $z = 4$. Applying the principle of deformation the integral I becomes

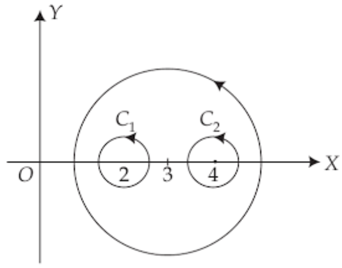


Fig. 15.20

$$I = \oint_C \frac{z+1}{z(z-2)(z-4)^3}\,dz$$

$$= \oint_{C_1}\left[\frac{z+1}{z(z-4)^3}\right]\frac{dz}{z-2} + \oint_{C_2}\left[\frac{z+1}{z(z-2)}\right]\frac{dz}{(z-4)^3} = I_1 + I_2, \text{ say.}$$

Now, $I_1 = \oint_{C_1}\left[\frac{z+1}{z(z-4)^3}\right]\frac{dz}{z-2} = 2\pi i\left[\frac{z+1}{z(z-4)^3}\right]_{z=2} = -\frac{3\pi i}{8}$, using Cauchy's integral formula.

Similarly, $I_2 = \oint_{C_2}\left[\frac{z+1}{z(z-2)}\right]\frac{dz}{(z-4)^3} = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left[\frac{z+1}{z(z-2)}\right]_{z=4} = \frac{23\pi i}{64}$

Therefore, $I = \frac{-3\pi i}{8} + \frac{23\pi i}{64} = -\frac{\pi i}{64}$.

Example 15.19: If $F(a) = \oint_C \frac{4z^2+z+5}{z-a}\,dz$, where C: $(x/2)^2 + (y/3)^2 = 1$, taken in counter-clockwise sense, then find $F(3.5)$, $F(i)$, $F'(-1)$ and $F''(-i)$.

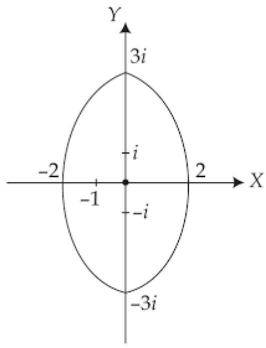


Fig. 15.21

Solution: We have, $F(3.5) = \oint_C \frac{4z^2+z+5}{z-3.5}\,dz$

The integrand $\frac{4z^2+z+5}{z-3.5}$ is analytic everywhere except at the point $z = 3.5$ which lies outside the ellipse $(x/2)^2 + (y/3)^2 = 1$, as shown in Fig. 15.21. Therefore, it is analytic everywhere within C and hence by Cauchy's integral theorem $F(3.5) = 0$.

Next the numerator $f(z) = 4z^2 + z + 5$ of the integrand is analytic everywhere in C and $a = i, -1$ and $-i$ all lie within C. Therefore by Cauchy's integral theorem, $f(a) = \frac{1}{2\pi i}\oint_C \frac{4z^2+z+5}{z-a}\,dz$, which gives

$$\oint_C \frac{4z^2+z+5}{z-a}\,dz = 2\pi i f(a) = 2\pi i[4a^2 + a + 5]$$

Hence $F(a)= 2\pi i[4a^2 + a + 5]$, which implies

$$F'(a) = 2\pi i[8a + 1] \text{ and } F''(a) = 16\pi i.$$

Thus, $$F(i) = 2\pi(i + 1),\ F'(-1) = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

15.5 CONVERSE OF CAUCHY'S INTEGRAL THEOREM: MORERA'S THEOREM. CAUCHY'S INEQUALITY. LIOUVILLE'S THEOREM

In this section, we prove the converse of the Cauchy's integral theorem called the Morera's theorem. Two other results, Cauchy's inequality and Lioville's theorem associated with the complex integration are also discussed.

Theorem 15.8 (Morera's theorem): *If f(z) is continuous in a simply connected domain D and if* $\oint_C f(z)dz = 0$ *for every closed path C in D, then f(z) is analytic in D.*

Proof. Let z be an arbitrary point in D and z_0 be a fixed point in D. If $f(z)$ is continuous and its integral around any closed path C in D is zero, then the definite integral of $f(z)$, refer Theorem 15.5, is defined by

$$F(z) = \int_{z_0}^{z} f(z^{\bullet})dz^{\bullet}$$

and further $F(z)$ is analytic in D with $F'(z) = f(z)$. Now the analyticity of $F(z)$ in D implies the analyticity of $F'(z)$ and hence that of $f(z)$ in D. This proves the converse of Cauchy's theorem.

Next, using the generalized Cauchy's integral formula (15.19), we find a bound for $f^{(n)}(z_0)$; a result known as *Cauchy's inequality* given as follows:

Theorem 15.9 (Cauchy's inequality): *If f(z) is analytic within and on a circle C:* $|z - z_0| = r$ *and* $|f(z)| \le M$ *on C, then*

$$|f^{(n)}(z_0)| \le \frac{M(n!)}{r^n} \qquad \ldots(15.26)$$

Proof. To prove this, choose the contour C in the generalized Cauchy's integral formula (15.19), a circle $|z - z_0| = r$ and apply the ML-inequality with $|f(z)| \le M$ on C, we obtain

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z - z_0)^{n+1}}dz\right| \le \frac{n!}{2\pi} M.\frac{1}{r^{n+1}} 2\pi r$$

or, $$|f^{(n)}(z_0)| \le \frac{n!M}{r^n},$$

This proves the Cauchy's inequality (15.26).

For $n = 0$, the Cauchy's inequality gives $|f(z_0)| \le M$, where z_0 is the centre of circle C and M is the bound of $f(z)$ on C. From this, there follows another important result called the '*Maximum-minimum principle*' stated next.

Theorem 15.10 (Maximum-minimum principal): *If $f(z)$ is analytic within and on a simple closed curve C and $f(z)$ is not a constant, then the maximum and minimum value of $|f(z)|$ occurs on the boundary of C.*

Next using the Cauchy's inequality we prove another important result on *entire functions*, *the functions which are analytic everywhere*, called the Liouville's theorem.

Theorem 15.11 (Liouville's theorem): *If $f(z)$ is entire and bounded for all z in the complex plane, then $f(z)$ must be constant.*

Proof. Since $f(z)$ is bounded, say $|f(z)| \le M$ for all z. The Cauchy's inequality (15.26) for $n = 1$, gives

$$|f'(z)| \le M/\mathrm{r} \qquad \ldots(15.27)$$

Further $f(z)$ is analytic for every z in the complex plane, so we can take r in (15.27) as large as we please and conclude that $|f'(z_0)| \to 0$. Since z_0 is arbitrary, thus $f'(z) = 0$ for all z and therefore $f(z)$ is constant.

Example 15.20: If $f(z) = e^z$, then find a bound on $f^{(n)}(0)$.

Solution: The function $f(z)$ is analytic for all z in the finite complex plane. Consider a unit circle $|z| = 1$ about the point $z = 0$, then

$$|f(z)| = |e^z| = e^x \le e$$

for all z on C and hence by Cauchy's inequality

$$|f^{(n)}(0)| \le \frac{n!M}{r} = \frac{(n!)e}{1} = (n!)e$$

Example 15.21: By direct calculations verify the maximum-minimum principle for $f(z) = \sin z$ in the domain D defined by $0 \le x \le \pi$ and $0 \le y \le 1$ and find bounds on $|\sin z|$ inside D.

Solution: The function $f(z) = \sin z$ is analytic for all z and the domain D is bounded. We have

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

and thus,

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y.$$

Differentiating it w.r.t. x and y, we obtain respectively

$$\begin{aligned}\frac{\partial}{\partial x}|\sin z|^2 &= 2\sin x \cos x \cosh^2 y - 2\cos x \sin x \sinh^2 y \\ &= \sin 2x(\cosh^2 y - \sinh^2 y) = \sin 2x,\end{aligned}$$

and,
$$\begin{aligned}\frac{\partial}{\partial y}|\sin z|^2 &= 2\sin^2 x \cosh y \sinh y + 2\cos^2 x \sinh y \cosh y \\ &= \sinh 2y\,[\sin^2 x + \cos^2 x] = \sinh 2y.\end{aligned}$$

The extreme values of $|\sin z|^2$, and hence that of $|\sin z|$ will occur at those points of D where both of these derivatives vanish simultaneously.

Now $\sin 2x = 0 \Rightarrow x = \pi/2$, but $\sinh 2y \neq 0$ for $0 < y < 1$, so $|\sin z|^2$ and hence $|\sin z|$ has neither maxima nor minima in D.

On the boundary $x = 0$ of D, as shown in Fig. 15.22,

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} = \sinh y.$$

It has a minimum value 0 at (0, 0) and a maximum value sinh1 at (0, 1).

Next on the boundary $x = \pi$ of D

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} = \sinh y$$

has a minimum value 0 at $(\pi, 0)$ and a maximum value sinh1 at $(\pi, 1)$.

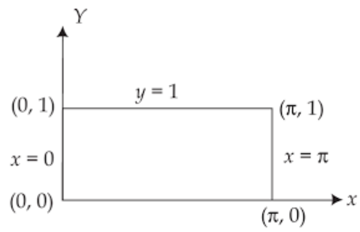


Fig. 15.22

Proceeding similarly, we find that on the boundary $y = 0$ of D, $|\sin z|$ has two minima of 0 at (0, 0) and $(\pi, 0)$ and a maximum of 1 at $(\pi/2, 0)$, and on the boundary $y = 1$ of D, $|\sin z|$ has two minima equal to sinh 1 at (0, 1) and $(\pi, 1)$ and a maximum of $\sqrt{1 + \sinh^2 1}$ at $(\pi/2, 1)$.

Thus the smallest value of $|\sin z|$ on the boundary of D is zero, and the largest value is $\sqrt{1+\sinh^2 1}$. This verifies maximum-minimum principle, and hence $0 < |\sin z| < \sqrt{1+\sinh^2 1}$ for all $z = x + iy$ in D.

EXERCISE 15.3

1. Evaluate the integral $\oint_C \frac{dz}{z^2+9}$, where C is

(a) $|z - 3i| = 4$ (b) $|z + 3i| = 2$ (c) $|z| = 5$

taken in counter-clockwise sense.

2. Evaluate the following integrals, the contour C being taken in counter-clockwise sense

(a) $\oint_C \frac{z^2+1}{z(2z+1)} dz$; C: $|z| = 1$ (b) $\oint_C \frac{e^{2z}}{(z-1)(z-2)} dz$; C: $|z| = 3$

(c) $\oint_C \frac{\cos \pi z}{z^2 - 1} dz$; C the rectangle with vertices $2 \pm i$, $-2 \pm i$

(d) $\oint_C \frac{\cosh (z^2 - \pi i)}{z - \pi i} dz_i$; C the rectangle with vertices ± 4 and $\pm 1 + 4i$

(e) $\oint_C \frac{z-3}{z^3+z} dz$; C: $|z| = 2$

(f) $\oint_C \frac{\sin(iz)}{z^2+1}dz$; C the triangle with vertices at $1-2i$, $-1-2i$ and $2i$.

3. Evaluate the following integrals, the contour C being taken in counter-clockwise sense

(a) $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}dz$; C: $|z|=1$ (b) $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$; C: $|z|=4$

(c) $\oint_C \frac{1}{(z^2+4)^2}dz$; C: $|z-i|=2$ (d) $\oint_C \frac{e^z}{z^2(z+1)^3}dz$; C: $|z|=2$

(e) $\oint_C \left[\frac{e^{iz}}{z^3}+\frac{z^4}{(z+i)^2}\right]dz$; C: $|z|=2$ (f) $\oint_C \frac{\sin z}{z^m}dz$; C: $|z|=1$, $m=2, 3, 4, \ldots$

4. Evaluate the contour integral of $f(z)=\frac{(2+i)\sin z^4}{(z+4)^2}$ over Γ, any closed path enclosing -4.

5. Let f be differentiable on a domain D containing a closed path Γ and all points enclosed by Γ. Prove that $\oint_\Gamma \frac{f'(z)}{z-a}dz = \oint_\Gamma \frac{f(z)}{(z-a)^2}dz$ for any 'a' enclosed by Γ.

6. If the function $f(z)$ is analytic inside and on a simple closed curve C containing the point $z=a$ inside it, then show that

$$f^{(n)}(a) = \frac{n!}{2\pi}\int_0^{2\pi} e^{-in\theta} f(a+e^{i\theta})d\theta,\ \ \text{n} = 0, 1, 2, \ldots.$$

7. Using the Cauchy's inequality find a bound on $f^{(n)}(0)$, when $f(z)=e^{3z}$ and C: $|z|=1$.

8. Verify the maximum/minimum principle for the function $f(z)=e^z$ in the domain $-1 \le x \le 1$, $-2 \le y \le 2$ and place bounds on $|e^z|$ inside the given domain.

ANSWERS

Exercise 15.1 (p. 67)

1. (a) $\frac{1}{3}(2+11)i$ (b) $\frac{-86}{3}-6i$ (c) $\frac{16}{3}i$

2. (a) $\frac{5}{2}(2-i)$ (b) $\frac{1}{3}(14+11i)$ 3. 30 4. $\frac{-152}{15}-12i$

5. $\left(\pi-\frac{1}{2}\sinh 2\pi\right)i$ 6. $2\,i\sinh\pi$.

Exercise 15.2 (p. 77)

2. (a) Yes !, 0 (b) Yes !, 0 (c) No !, $5\pi i$ (d) No!, 0
3. $8\pi i$
4. (a) Yes!, 0 (b) No, 0 (c) No, $-\pi$
5. (a) 0, (b) 0 (c) 0
6. (a) $4\pi i$ (b) $4\pi i$ (c) 0
9. $(e^{-2} - e^{-6})/2$
10. (a) $-\dfrac{i}{3}\sinh 3$ (b) $(e^{-9} - 1)/2$ (c) $[1 - \cos(-3 + 4i)]/2$
    (d) $\dfrac{1}{3}(\pi i + 1)^3 + \dfrac{1}{2}\sinh 2$ (e) $\pi^2/32$ (f) 0

Exercise 15.3 (p. 87)

1. (a) $\pi/3$ (b) $-\pi/3$ (c) 0
2. (a) $-\dfrac{\pi i}{2}$ (b) $2\pi i(e^4 - e^2)$ (c) 0
   (d) $-2\pi i \cosh \pi^2$ (e) 0 (f) $-2\pi \sin 1$
3. (a) πi (b) i/π (c) $\pi/16$
   (d) $(11e^{-1} - 4)\pi i$ (e) $-\pi(8 + i)$ (f) $\dfrac{2\pi i}{(m-1)!}\sin\left[(m-1)\dfrac{\pi}{2}\right]$
4. $-512\pi(1 - 2i)\cos(256)$ 7. $e^3(n!)$ 8. $\dfrac{1}{e} < |e^z| < e.$

16 CHAPTER
Taylor Series, Laurent Series and The Residue Theorem

"Power series in general, and Taylor series in particular, are direct generalizations of the power and Taylor series in reals. The Laurent series, a series of positive and negative powers of $(z - z_0)$, represents an extension of the Taylor series which is no longer applicable when an expansion of $f(z)$ is required about a singularity z_0 and is used to classify points at which $f(z)$ is not analytic. Residue of $f(z)$ at z_0 is the coefficient of $(z - z_0)^{-1}$ in the Laurent series expansion of $f(z)$ about $z = z_0$. Residues are used to compute contour integration and certain complicated real integrals as well".

16.1 COMPLEX SERIES AND CONVERGENCE TESTS

In this section we define complex series and discuss their convergence and divergence. Also we consider the concept of absolute, conditional and that of uniform convergence.

16.1.1 Complex Series and Their Convergence

By a *complex series* we mean a series of the form

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \ldots, \qquad \ldots(16.1)$$

where the z_n's are complex numbers. As in the real case, we say that the series (16.1) *converges* if the limit of the sequence of *partial sums*, $S_n = z_1 + z_2 + \ldots + z_n$ exists as $n \to \infty$. That is, the series converges to S if to each number $\in > 0$, no matter how small, there exists an integer $N(\in)$ such that $|S_n - S| < \in$ for all $n > N$; S is called the *sum* of the series.

A series that is not convergent is called a *divergent series*.

Further the series Σz_n can be expressed in terms of two real series for $z_n = x_n + iy_n$. Thus, Σz_n converges and has the sum $S = U + iV$ if, and only if the series Σx_n converges and has the sum U and the series Σy_n converges and has the sum V.

Convergence or divergence tests in complex series are practically the same as in case of reals.

Cauchy's convergence principle for complex series. *An infinite series Σz_n is convergent if, and only if for every $\in > 0$, no matter how small, we can find $N(\in)$ such that*

$$|z_{n+1} + z_{n+2} + \ldots + z_{n+m}| < \in, \text{ for every } n > N,\ m = 1, 2, \ldots \quad \ldots(16.2)$$

In case we set $m = 1$, (16.2) gives that to each $\in > 0$, no matter how small, there corresponds an integer $N(\in)$ such that $|z_{n+1}| < \in$ for all $n > N$, which is equivalent to saying that $z_{n+1} \to 0$ (or, for that matter $z_n \to 0$) as $n \to \infty$. Thus, we have the following result:

Theorem 16.1 (A necessary condition for convergence): *If an infinite series Σz_n converges, then $\lim\limits_{n \to \infty} z_n = 0$.*

We must note that *it is only the necessary condition for convergence but not sufficient*. For example, the series $1 + 1/2 + 1/3 + \ldots$ satisfies this condition but is divergent one. Thus this result can be applied only to prove the non-convergence of the series.

Absolute convergence. *A series Σz_n is said to be 'absolutely convergent' if the series of the absolute values of the terms $\Sigma|z_n| = |z_1| + |z_2| + \ldots$ is convergent.*

If the series Σz_n converges but $\Sigma|z_n|$ diverges, then the series Σz_n is called *conditionally convergent*. For example, the series $1 - 1/2 + 1/3 - 1/4 + \ldots$ is only conditionally convergent.

If a series is absolutely convergent, then it is convergent also. But the divergence of the series $\Sigma|z_n|$ does not imply the divergence of the series Σz_n.

16.1.2 Tests for Convergence and Divergence

We discuss a few tests for convergence of the series.

I. Comparison test: *For a given series Σz_n if we can find a convergent series Σb_n of non-negative real terms such that $|z_n| \le b_n$, $n = 1, 2, \ldots$, then the series Σz_n is also convergent, even absolutely.*

The result follows from the Cauchy's convergence principle for series.

For comparison, normally we use the geometric series

$$\Sigma x^n = 1 + x + x^2 + \ldots,$$

which converges to the sum $1/(1 - x)$ for $|x| < 1$ and diverges for $|x| \ge 1$.

II. Ratio test: *If for a series Σz_n, $z_n \ne 0$, $(n = 1, 2,\ldots)$, $\lim\limits_{n \to \infty} |z_{n+1}/z_n| = l$, then the series Σz_n converges absolutely, if $l < 1$ and diverges absolutely,* if $l > 1$.

In case $l = 1$, or if the limit does not exist, then no conclusion is drawn.

An other powerful test to test the convergence of the complex series is the root test.

III. Root test: *If for a series Σz_n, $\lim\limits_{n \to \infty} (z_n)^{1/n} = l$, then the series Σz_n converges absolutely, if $l < 1$ and diverges absolutely, if $l > 1$.*

In case $l = 1$, or if the limit does not exist, then no conclusion is drawn.

Example 16.1: Test for convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \ldots$$

Solution: Consider $|z_n| = \left|\frac{i^n}{n!}\right| = \frac{1}{n!} < \frac{1}{2^n}$ for all $n \geq 4$. Further the series $\sum_{n=0}^{\infty} 1/2^n$ is a geometric series with common ratio $r = \frac{1}{2} < 1$, and hence, is a convergent series. Thus it follows from the comparison test that the given series is also convergent.

Example 16.2: Test for convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!}.$$

Solution: Consider $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(100+75i)^{n+1}}{(n+1)!} \cdot \frac{n!}{(100+75i)^n}\right| = \left|\frac{100+75i}{n+1}\right| = \frac{125}{n+1}$ which tends to zero as $n \to \infty$. Hence by ratio test the series is convergent.

Example 16.3: Test for convergence or divergence of the series $\Sigma e^{-(2+3i)n}$

Solution: Consider $\left|\frac{z_{n+1}}{z_n}\right| = |e^{-(2+3i)}| = |e^{-2}.e^{-3i}| = e^{-2} < 1$

Hence by ratio test the series is convergent.

Example 16.4: Test for convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{3n-2}{np+1}\right)^n (3-4i)^n$$

Solution: Here, $|z_n| = \left(\frac{3n-2}{np+1}\right)^n |3-4i|^n = \left(\frac{3n-2}{np+1}\right)^n 5^n = \left(\frac{15n-10}{np+1}\right)^n$

Consider $\lim_{n\to\infty} |z_n|^{1/n} = \lim_{n\to\infty} \frac{15n-10}{np+1} = \frac{15}{p}$

Hence by root test the given series is convergent for $p > 15$ and divergent for $p < 15$. For $p = 15$ we can verify that $\lim_{n\to\infty} z_n \neq 0$ and hence the series is divergent for $p = 15$.

16.1.3 Uniform Convergence of Series of Functions

In general, the terms of an infinite series may be functions of z rather than constants, that is, $\Sigma f_n(z) = f_1(z) + f_2(z) + \ldots$. Then the set of all points in the z-plane for which the series converges is called the *region of convergence* of the series.

Let $\Sigma f_n(z)$ be a series of single valued complex functions defined in a domain D and $S_n(z) = f_1(z) + \ldots + f_n(z)$ be the nth partial sum. Consider the sequence $\{S_n(z)\}$ of partial sums. If at a point $z = z_0$ in

D, the sequence $\{S_n(z)\}$ of partial sums converges to $f(z_0)$, then we say that the series $\Sigma f_n(z_0)$ converges to $f(z_0)$. This convergence is called *'pointwise convergence'*.

If for each $z \in D$, the sequence $\{S_n(z)\}$ of partial sums converges to $f(z)$, then we say that the series $\Sigma f_n(z)$ *converges uniformly* to $f(z)$, that is, for every $\in > 0$, no matter how small, there exists an $N(\in)$ independent of z, such that

$$|S_n(z) - f(z)| < \in, \text{ for all } n > N(\in).$$

A series which is uniformly convergent is also pointwise convergent.

Next we state the sufficient condition for a given series to be uniformly convergent.

Weierstrass's M-test. *Let $\Sigma f_n(z)$ be an infinite series of single valued complex functions defined in a domain D and let $\{M_n\}$ be a sequence of positive terms, where $|f_n(z)| \le M_n$ for $n = 1, 2, \ldots$ and for all $z \in D$. If the series ΣM_n is convergent, then the series $\Sigma f_n(z)$ is uniformly and absolutely convergent.*

Example 16.5: Show that $\sum \frac{z^n - 1}{n^2 + |z|^2}$ converges uniformly for $|z| < 1$.

Solution: We have, $|f_n(z)| = \left|\frac{z^n - 1}{n^2 + |z|^2}\right| \le \frac{|z|^n + 1}{n^2 + |z|^2} < \frac{2}{n^2}$, for all z in $|z| < 1$.

Since, the series $\sum \frac{1}{n^2}$ is convergent, the given series is uniformly convergent by Weierstrass's M-test.

Remark: There is no relation between absolute and uniform convergence. There are series that converge absolutely but not uniformly and other that converge uniformly but not absolutely.

EXERCISE 16.1

Test for convergence or divergence of the following series:

1. $\sum_{n=1}^{\infty} n^2 \left(\frac{i}{2}\right)^n$ **2.** $\sum_{n=1}^{\infty} \frac{(3i)^n n!}{n^n}$ **3.** $\sum_{n=1}^{\infty} \frac{\cos(2n - 3i)}{n^p}$

4. $\sum \left(\frac{3}{4}i\right)^n n^3$ **5.** $\sum_{n=1}^{\infty} \frac{n}{(2+i)^n}$ **6.** $\sum_{n=1}^{\infty} e^{-in}$

7. Show that the series $\Sigma (\sin nz)/n^2$ is uniformly convergent in $|z| \le 1$.

8. Show that the geometric series $1 + z + z^2 + \ldots$ is (a) uniformly convergent in any closed disk $|z| \le r < 1$, (b) not uniformly convergent in its whole disk of convergence $|z| < 1$.

9. Show that the series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots$ converges absolutely but not uniformly.

10. Show that the series $\frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^2+3} - \ldots$ converges uniformly but not absolutely.

16.2 POWER SERIES REPRESENTATIONS

Power series are the most important series in the study of complex analysis since we shall observe that their sums are analytic functions and also every analytic function can be represented by a power series. In this section we discuss power series, their convergence and power series representations of the analytic functions.

16.2.1 Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \qquad \dots(16.3)$$

is called a *'power series'* about $z = z_0$. Here $a_0, a_1, \dots$ are real or complex constants called the *coefficients* of the series, and z_0 again a constant, real or complex, is called the *center of the series*. For $z_0 = 0$, we obtain a power series in powers of z given as

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots .$$

Convergence of a power series: We note that a power series always converges at the point $z = z_0$, since for $z = z_0$ the series reduces to a constant a_0. In general the series (16.3) may converge in a disk with center z_0, or in the whole z-plane, or only at z_0. For example, refer Example 16.6,

(a) the geometric series Σz^n converges absolutely if $|z| < 1$;

(b) the power series $\Sigma z^n/n! = 1 + z + z^2/2! + z^3/3! + \dots$ converges absolutely for every z; while

(c) the series $\Sigma n! z^n = 1 + z + 2z^2 + 6z^3 + \dots$, converges only at $z = 0$.

We have the following result for the convergence of series of the form (16.3).

Theorem 16.2 (Convergence of a power series):

(a) Every power series converges at its center z_0.

(b) If the power series converges at a point $z = z_1 \neq z_0$, then it converges absolutely for all z in the disk $|z - z_0| < |z_1 - z_0|$.

(c) If the power series diverges at a point $z = z_2$, then it diverges for all z farther away from z_0 than z_2, that is, for all z such that $|z - z_0| > |z_2 - z_0|$.

Proof.

(a) Obviously for $z = z_0$ the series (16.3) sums to simply a_0.

(b) The convergence at the fixed point $z = z_1$ implies that $a_n(z_1 - z_0)^n \to 0$ as $n \to \infty$. Thus, there exists a finite positive constant M such that $|a_n(z_1 - z_0)^n| \le M$ for every $n = 0, 1, 2, \dots$

Now $|a_n(z - z_0)^n| = |a_n||z - z_0|^n = |a_n||z_1 - z_0|^n . \left|\dfrac{z - z_0}{z_1 - z_0}\right|^n \le M\left|\dfrac{z - z_0}{z_1 - z_0}\right|^n$

For $|z - z_0| < |z_1 - z_0|$, the series $M \sum \left|\frac{z - z_0}{z_1 - z_0}\right|^n$ is a convergent geometric series with common ratio less than 1. Therefore the series $\Sigma a_n(z - z_0)^n$ converges absolutely for all z in $|z - z_0| < |z_1 - z_0|$.

(c) The result follows simply from (b), since convergence at a point z_3, farther away from z_0 than z_2, would imply convergence at z_2 which is a contradiction.

Radius of convergence: Let R be the radius of the smallest circle with center at z_0 that contains all the points at which the power series (16.3) is convergent. Then the series is convergent for all z for which $|z - z_0| < R$ and diverges for all z for which $|z - z_0| > R$. The real number R is called the '*radius of convergence*' and the circle $|z - z_0| = R$ is called the '*circle of convergence*' of the power series (16.3).

If $R = 0$ the series is convergent only at the point $z_0 = 0$, and if $R = \infty$ the series is convergent for all z.

On the circle of convergence, the series may converge at some, or all, or none of the points. For example, in the three cases listed below the radius of convergence is $R = 1$ but

(a) The series $\Sigma z^n/n^2$ converges everywhere on R, since $\Sigma z^n/n^2 = \Sigma 1/n^2$ on $|z| = 1$;

(b) The series $\Sigma z_n/n$ converges at $z = -1$ but diverges at $z = 1$;

(c) The series Σz^n diverges everywhere on $|z| = 1$.

16.2.2 Test for the Convergence of a Power Series

The convergence of a power series $\Sigma a_n(z - z_0)^n$ can be determined by the application of the ratio test. Let z_n denote the nth term of the power series, then we obtain

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{a_{n+1}}{a_n}(z - z_0)\right| = \left|\frac{a_{n+1}}{a_n}\right||z - z_0| \text{ and, therefore,}$$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L^\bullet |z - z_0|, \text{ where } L^\bullet = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Thus by ratio test, the series converges, if $L^\bullet |z - z_0| < 1$, or $|z - z_0| < 1/L^\bullet$, provided $L^\bullet \neq 0$, and diverges, if $|z - z_0| > 1/L^\bullet$, and the *radius of convergence* of the power series (16.3) is

$$R = \frac{1}{L^\bullet} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

We can also use the Cauchy's root test and write $L^\bullet = \lim_{n\to\infty} |a_n|^{1/n}$.

Example 16.6: Find the region of convergence for the following power series:

(a) $\sum_{n=0}^{\infty} z^n$ (b) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ (c) $\sum_{n=0}^{\infty} n!z^n$

Solution:

(a) Here $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = |z|$. Hence by ratio test the series converges for all $|z| < 1$ and radius of convergence is $R = 1$.

(b) Here, $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 0 < 1$. Hence by ratio test the series converges for all z and radius of converges is $R = \infty$.

(c) Here, $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} |(n+1)z| \to \infty$. Thus the series is divergent for all z except at $z = 0$, its center, hence the radius of convergence is $R = 0$.

Example 16.7: Find the radius of convergence, region of convergence and the circle of convergence of the following power series

(a) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$ (b) $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^{n^2} z^n$ (c) $\sum_{n=1}^{\infty} \frac{n(5+2i)^n}{3^n}(z-1)^n$

Solution: (a) We have

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n)!}{(n!)^2}\cdot\frac{[(n+1)!]^2}{(2n+2)!}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$$

Thus the series converges in the region $|z-3i| < 1/4$. The circle of convergence is $|z-3i| = 1/4$ with center $z_0 = 3i$.

(b) $$R = \frac{1}{\lim\limits_{n\to\infty}|a_n|^{1/n}} = \lim_{n\to\infty}\left[1+\frac{2}{n}\right]^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^{-2} = e^{-2}$$

Thus, the series converges in the region $|z| < e^{-2}$. The circle of convergence is $|z| = e^{-2}$ with center $z = 0$.

(c) $$R = \frac{1}{\lim\limits_{n\to\infty}|a_n|^{1/n}} = \lim_{n\to\infty}\frac{3}{|5+2i|}\cdot\frac{1}{n^{1/n}} = \frac{3}{|5+2i|}\lim_{n\to\infty}(1/n^{1/n}) = \frac{3}{\sqrt{29}}$$

Thus, the series converges in the region $|z-1| < 3/\sqrt{29}$. The circle of convergence is $|z| = 3/\sqrt{29}$ with center $z = 0$.

16.2.3 Function Represented by Power Series

Using Cauchy's convergence principle for series we can check very easily that the power series is uniformly convergent within its circle of convergence. Let the power series $\Sigma a_n(z-z_0)^n$ converge to the function $f(z)$. To simplify the formulae, without loss of generality, we can take $z_0 = 0$. Thus, if the power series $\Sigma a_n z^n$ has a non-zero radius of convergence R with sum, say $f(z)$, then we write

$$f(z) = \Sigma a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots, \ |z| < R$$

and we say that $f(z)$ is represented by the power series. In fact, *a power series* $\Sigma a_n z^n$ *represents an analytic function within its circle of convergence.*

For example, the geometric series Σz^n represents the function $1/(1 - z)$ which is analytic within its circle of convergence $|z| = 1$.

Further, this power series representation is unique in the sense that a function $f(z)$ cannot be represented by two different power series with the same center.

16.2.4 Operations on Power Series

We define the following operations:

(a) **Termwise addition or subtraction:** If $f(z) = \Sigma a_n z^n$ and $g(z) = \Sigma\, b_n z^n$ are two power series with radii of convergence R_1 and R_2 respectively, then the termwise addition or subtraction of the series is defined as

$$f(z) \pm g(z) = \sum (a_n \pm b_n) z^n$$

The radius of convergence is equal to the minima of R_1 and R_2.

(b) **Termwise multiplication:** If $f(z) = \Sigma a_n z^n$ and $g(z) = \Sigma\, b_n z^n$ are two power series with radii of convergence R_1 and R_2 respectively, then the termwise multiplication, called the '*Cauchy product*', of these series is defined as

$$f(z)\,g(z) = \sum_n \left(\sum_{r=0}^{n} a_r b_{n-r} \right) z^n = \sum c_n z^n, \text{ where } c_n = \sum_{r=0}^{n} a_r b_{n-r}\,.$$

The radius of convergence is equal to the minima of R_1 and R_2.

(c) **Termwise differentiation:** Termwise differentiation of a power series is permissible within its circle of convergence and the resultant series is called the '*derived series*'. It is given as

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \ldots$$

The radius of convergence of the derived series is the same as the radius of convergence of the original series.

(d) **Termwise integration:** Termwise integration of a power series is permissible within its circle of convergence. It is given as

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \ldots\,.$$

The radius of convergence of the integrated series is the same as the radius of convergence of the original series.

EXERCISE 16.2

Find the center and radius of convergence of the following power series:

1. $\sum_{n=0}^{\infty} (n+1)(z+1)^n$
2. $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$
3. $\sum_{n=0}^{\infty} (n+2i)^n z^n$
4. $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}(z+2-i)^n$
5. $\sum_{n=1}^{\infty} \left(1-\frac{\pi}{n}\right)^{n^2} z^n$
6. $\sum_{n=0}^{\infty} z^{n!}$
7. $\sum_{n=4}^{\infty} e^n(z+i)^n$
8. $\sum_{n=0}^{\infty} e^{in} z^n$
9. $\sum_{n=1}^{\infty} \frac{2^{n-1} z^{2n-1}}{(4n-3)^2}$
10. $\sum_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{z}{2}\right)^n$
11. $\sum_{n=1}^{\infty} i^n z^n$
12. $\sum_{n=1}^{\infty} \left(\frac{1+2ni}{n+2i}\right)^n z^n$
13. Using Cauchy's product of two power series show that if

$f(z) = \sum \frac{z^n}{n!}$ for all z, then $[f(z)]^2 = f(2z)$.

16.3 TAYLOR AND MACLAURIN SERIES

In the preceding section we have observed that the sum of a power series with positive radius of convergence is an analytic function. Here we will explore that the converse of this result is also true, that is, every given analytic function $f(z)$ which is analytic inside the region $|z - z_0| < R$ can be expressed as a power series inside this region. This series is called the *Taylor series* of the function $f(z)$, and is the complex analogous of the real Taylor series.

Theorem 16.3 (Taylor series): *A function $f(z)$ which is analytic inside a circle $|z - z_0| = R$ may be represented inside the circle as a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n; \quad a_n = \frac{1}{n!}f^{(n)}(z_0),$$

called the Taylor expansion of $f(z)$ about the point $z = z_0$.

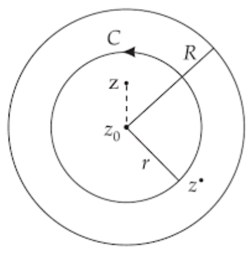


Fig. 16.1

Proof. Let z be any arbitrary point inside the circle $|z - z_0| = R$. Draw a circle with centre at z_0 of radius $r < R$ enclosing the point z and let $z^\bullet$ be any point on the circle $|z - z_0| = r$, as shown in Fig. 16.1. Consider

$$\frac{1}{z^\bullet - z} = \frac{1}{(z^\bullet - z_0) - (z - z_0)} = \frac{1}{(z^\bullet - z_0)}\left[1 - \frac{z-z_0}{z^\bullet - z_0}\right]^{-1} \quad ...(16.4)$$

Since $\left|\dfrac{z - z_0}{z^{\bullet} - z_0}\right| < 1$, expanding the right hand side of (16.4) in binomial series, we obtain

$$\frac{1}{z^{\bullet} - z} = \frac{1}{z^{\bullet} - z_0}\left[1 + \frac{z - z_0}{z^{\bullet} - z_0} + \left(\frac{z - z_0}{z^{\bullet} - z_0}\right)^2 + \ldots + \left(\frac{z - z_0}{z^{\bullet} - z_0}\right)^n\right] + \frac{1}{z^{\bullet} - z}\left(\frac{z - z_0}{z^{\bullet} - z_0}\right)^{n+1} \quad \ldots(16.5)$$

Now, since $f(z)$ is analytic inside the region $|z^{\bullet} - z_0| = r$ and z is an interior point of this, by Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^{\bullet})}{z^{\bullet} - z}\,dz^{\bullet}, \quad \ldots(16.6)$$

where C: $|z^{\bullet} - z_0| = r < R$, taken in counter-clockwise sense.

Inserting (16.5) in (16.6) yields

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^{\bullet})}{z^{\bullet} - z_0}\,dz^{\bullet} + \frac{z - z_0}{2\pi i}\oint_C \frac{f(z^{\bullet})}{(z^{\bullet} - z_0)^2}\,dz^{\bullet}$$

$$+ \frac{(z - z_0)^2}{2\pi i}\oint_C \frac{f(z^{\bullet})}{(z^{\bullet} - z_0)^3}\,dz^{\bullet} + \ldots + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(z^{\bullet})}{(z^{\bullet} - z_0)^{n+1}}\,dz^{\bullet} + R_n(z), \quad \ldots(16.7)$$

where $$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z^{\bullet})}{(z^{\bullet} - z_0)^{n+1}(z^{\bullet} - z)}\,dz^{\bullet}$$

is the *remainder term*.

Using Cauchy's generalized integral formula for derivatives we obtain from (16.7),

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!}f''(z_0) + \ldots + \frac{(z - z_0)^n}{n!}f^{(n)}(z_0) + R_n(z) \quad \ldots(16.8)$$

This is called *Taylor formula with remainder $R_n(z)$*.

Since $f(z)$ is analytic, it has derivatives of all orders, so n can be taken as large as desired. Taking $n \to \infty$, Eq. (16.8) becomes

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!}f''(z_0) + \ldots + \frac{(z - z_0)^n}{n!}f^{(n)}(z_0) + \ldots .$$

or, $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \ldots(16.9)$$

where $a_n = \dfrac{1}{n!}f^{(n)}(z_0)$, provided $\lim\limits_{n\to\infty} R_n(z) = 0$.

The series given by (16.9) is called the *Taylor series expansion of $f(z)$ about the point z_0*. Obviously the series, (16.9) will converge and represent $f(z)$ if, and only if $\lim\limits_{n\to\infty} R_n(z) = 0$, which we prove as follows

Since $|z^{\bullet} - z| \neq 0$ and $f(z)$ is analytic inside and on C, so it is bounded, and thus so is the function $\left|\frac{f(z^{\bullet})}{(z^{\bullet} - z)}\right|$, say $\left|\frac{f(z^{\bullet})}{(z^{\bullet} - z)}\right| \le M$, for all $z^{\bullet}$ on C. Also C has the radius $|z^{\bullet} - z_0| = r$ and length $2\pi r$. Hence, by ML-inequality, we have

$$|R_n(z)| = \frac{|z - z_0|^{n+1}}{2\pi}\left|\oint_C \frac{f(z^{\bullet})}{(z^{\bullet} - z_0)^{n+1}(z^{\bullet} - z)}dz^{\bullet}\right| \le \frac{|z - z_0|^{n+1}}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = Mr\left|\frac{z - z_0}{r}\right|^{n+1} \quad ...(16.10)$$

Now z lies inside C, thus $\left|\frac{z - z_0}{r}\right| < 1$, so the right side of (16.10) approaches zero as n tends to infinity. This proves the convergence of the Taylor series.

Since z is chosen as an arbitrary point inside C, so the convergence holds for every z inside C. Hence, the *convergence is uniform*.

Further the expansion (16.9) is unique, since otherwise, consider that

$$f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n, \quad ...(16.11)$$

where at least one of the coefficients $b_n \neq a_n$. Now, the power sereis (16.11) converges to $f(z)$ in the circular region $|z - z_0| < R$, and thus $b_n = \frac{1}{n!} f^{(n)}(z_0)$ which is same as a_n. This proves the uniqueness.

Remarks 1. The radius of convergence of Taylor series expansion of a function $f(z)$ is the distance between z_0 and the nearest singular point of $f(z)$. For example $1/(1 - z)$ has singularity at $z = 1$, its expansion, the geometric series has radius of convergence 1.

2. Since the Taylor's series representation of an analytic function is unique, we can obtain this representation directly expanding the function using binomial theorem for any index, whenever possible. For example, the expansion of $f(z) = 1/(1 - z)$ can be obtained simply as $(1 - z)^{-1} = 1 + z + z^2 + z^3 + ...$, $|z| < 1$.

3. Taylor's series are power series and conversly also, a power series with non-zero radius of convergence is the Taylor series of its sum.

4. The complex analytic functions can always be represented by power series of the form $\Sigma a_n(z - z_0)^n$, which is not true in general for real functions. There are real functions with derivatives of all orders but cannot be represented by a power series e.g., $f(x) = e^{-1/x^2} \neq 0$, and $f(0) = 0$ about $x = 0$.

Maclaurin's series: *A Maclaurin's series is a Taylor series with center $z_0 = 0$. Thus Maclaurin's series representation of $f(z)$ is*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where} \quad a_n = \frac{f^{(n)}(0)}{n!} \quad ...(16.12)$$

Next, we list some important power series expansions, which can be derived as in case of calculus of a real variable simply replacing x by z, since the coefficient formulae are the same.

(a) $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \ldots + \dfrac{z^n}{n!} + \ldots,\ |z| < \infty$

(b) $\dfrac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots,\ |z| < 1$

(c) $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots + (-1)^n z^n + \ldots,\ |z| < 1$

(d) $\text{Ln}\,(1 + z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \ldots + (-1)^{n+1}\dfrac{z^n}{n}\ \ldots,\ |z| < 1$

(e) $\text{Ln}\,(1 - z) = -z - \dfrac{z^2}{2} - \dfrac{z^3}{3} - \ldots - \dfrac{z^n}{n} - \ldots,\ |z| < 1$

(f) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots + (-1)^{n+1}\dfrac{z^{2n-1}}{(2n-1)!} + \ldots,\ |z| < \infty$

(g) $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \ldots + (-1)^n \dfrac{z^{2n}}{(2n)!} + \ldots,\ |z| < \infty$

(h) $\sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \ldots + \dfrac{z^{2n+1}}{(2n+1)!} + \ldots,\ |z| < \infty$

(i) $\cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \ldots + \dfrac{z^{2n}}{(2n)!} + \ldots,\ |z| < \infty$

Example 16.8: Expand $f(z) = 1/z$ about $z = 1$ in Taylor series. Obtain its radius of convergence.

Solution: For $z_0 = 1$, the Taylor series is $f(z) = \sum_{n=0}^{\infty} a_n(z-1)^n$, where $a_n = \dfrac{f^{(n)}(1)}{n!}$

Here, $f(z) = \dfrac{1}{z}$. Thus $f'(z) = -\dfrac{1}{z^2}, f''(z) = \dfrac{(-1)^2 2!}{z^3}, \ldots, f^{(n)}(z) = \dfrac{(-1)^n n!}{z^{n+1}}$

$f^{(n)}(1) = (-1)^n n!$, and hence $a_n = (-1)^n$. This gives

$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots + (-1)^n (z-1)^n \ldots$$

as the required series.

The radius of convergence is $R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} |-1| = 1.$

Alternatively, consider

$$\frac{1}{z} = \frac{1}{1+(z-1)} = [1 + (z-1)]^{-1} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots,$$

provided $|z-1| < 1$.

Example 16.9: Expand $f(z) = 2i/(4 + iz)$ about $z = -3i$ in Taylor series.

Solution: Consider

$$f(z) = \frac{2i}{4+iz} = \frac{2i}{4+i(z+3i)+3} = \frac{2i}{7+i(z+3i)} = \frac{2i}{7}\frac{1}{\left[1+\frac{i}{7}(z+3i)\right]} = \frac{2i}{7}\left[1+\frac{i}{7}(z+3i)\right]^{-1}$$

$$= \frac{2i}{7}\sum_{n=0}^{\infty}(-1)^n\left[\frac{i}{7}(z+3i)\right]^n, \text{ provided } \left|\frac{i}{7}(z+3i)\right| < 1, \text{ or } |z+3i| < 7.$$

It is the Taylor series expansion of $f(z) = 2i/(4 + iz)$ about the point $z_0 = -3i$ which converges in the region $|z + 3i| < 7$. Hence the radius of convergence is 7.

Example 16.10: Find the Taylor series expansion of $f(z) = 1/(1 - z)^3$ about the origin.

Solution: We know that

$$g(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

Differentiating it termwise, we obtain

$$g'(z) = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1}$$

Differentiating again

$$g''(z) = \frac{2}{(1-z)^3} = \sum_{n=2}^{\infty} n(n-1)z^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)z^n$$

Hence, $f(z) = \dfrac{1}{(1-z)^3} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(n+1)(n+2)z^n, \quad |z| < 1$

is the Taylor series expansion about $z = 0$.

Alternatively, consider

$$\frac{1}{(1-z)^3} = (1-z)^{-3} = 1 + 3z + \frac{3(4)}{2!}z^2 + \frac{3(4)(5)}{3!}z^3 + \ldots, \quad |z| < 1$$

$$= \frac{1}{2}[2 + (2)(3)z + (3)(4)z^2 + (4)(5)z^3 + \ldots] = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^n, \quad |z| < 1.$$

Example 16.11: Obtain the series expansion of $f(z) = \dfrac{1}{z^2+(1+2i)z+2i}$ about $z = 0$

Solution: We have

$$f(z) = \frac{1}{z^2+(1+2i)z+2i} = \frac{1}{(z+2i)(z+1)} = \frac{1}{(1-2i)}\left[\frac{1}{z+2i} - \frac{1}{z+1}\right]$$

Consider, $\dfrac{1}{z+2i} = \dfrac{1}{2i}\left[1+\dfrac{z}{2i}\right]^{-1} = \dfrac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{z}{2i}\right)^n$; $\left|\dfrac{z}{2i}\right| < 1$, or $|z| < 2$.

Also, $\dfrac{1}{z+1} = [1+z]^{-1} = \sum_{n=0}^{\infty}(-1)^n z^n$, $|z| < 1$.

Hence, $$f(z) = \frac{1}{1-2i}\left[\frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2i}\right)^n - \sum_{n=0}^{\infty}(-1)^n z^n\right]$$

$$= \frac{1}{1-2i}\left[\sum_{n=0}^{\infty}(-1)^n\left\{\left(\frac{1}{2i}\right)^{n+1} - 1\right\}z^n\right],$$

with radius of convergence, $R = \min.\{1, 2\} = 1$.

Example 16.12: Find the Taylor series expansion of

$$f(z) = \frac{2z^2+9z+5}{z^3+z^2-8z-12}$$

about the point $z = 1$.

Solution: We have,

$$f(z) = \frac{2z^2+9z+5}{z^3+z^2-8z-12} = \frac{2z^2+9z+5}{(z+2)^2(z-3)} = \frac{1}{(z+2)^2} + \frac{2}{(z-3)} \qquad ...(16.13)$$

Consider $\dfrac{1}{(z+2)^2} = \dfrac{1}{[3+(z-1)]^2} = \dfrac{1}{9}\left[1+\dfrac{z-1}{3}\right]^{-2}$

$$= \frac{1}{9}\left[1+(-2)\left(\frac{z-1}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{z-1}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{z-1}{3}\right)^2 + \ldots\right], \ |z-1| < 3,$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n, \ |z-1| < 3.$$

Also, $$\frac{1}{z-3} = \frac{1}{-2+(z-1)} = -\frac{1}{2}\left[1-\frac{z-1}{2}\right]^{-1} = -\frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{(z-1)}{2}\right]^n, \quad |z-1|<2$$

Substituting these in (16.13) yields

$$f(z) = \left[\frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n\right] = \sum_{n=0}^{\infty}\left[\binom{-2}{n}\frac{1}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n,$$

with radius of convergence $R = \min\{2, 3\} = 2$.

Example 16.13: Find Taylor series expansion of the function $f(z) = Ln\, z$ about the point $z = (-1 + i)$. Obtain its radius of convergence.

Solution: We have, $f(z) = Ln\, z = Ln\,[(-1+i) + \{z - (-1+i)\}] = Ln(-1+i) + Ln\left[1 + \dfrac{z-(-1+i)}{-1+i}\right]$

Using $$Ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{z} + \ldots$$

We obtain, $$f(z) = Ln(-1+i) + \frac{z-(-1+i)}{-1+i} - \frac{1}{2}\left(\frac{z-(-1+i)}{-1+i}\right)^2 + \frac{1}{3}\left(\frac{z-(-1+i)}{-1+i}\right)^3 + \ldots$$

as the Taylor series expansion about $(-1+i)$.

The series is convergent for $\left|\dfrac{z-(-1+i)}{-1+i}\right| < 1$, or $|z-(-1+i)| < \sqrt{2}$. Hence the radius of convergence is $\sqrt{2}$.

EXERCISE 16.3

Develop the following functions in a Taylor series about the given point as center. Find the radius of convergence.

1. $1/z, 2$ **2.** e^z, a **3.** $\sin z, \pi/2$

4. $\sinh z, \pi i/2$ **5.** $\cosh z, \pi i$ **6.** $Ln(2+iz), i$

Develop the follwing functions in a Maclaurin's series. Find the radius of convergence.

7. $1/(1+z^2)$ **8.** $\tan^{-1} z$ **9.** $\sin^2 z$

10. $\dfrac{z+2}{1-z^2}$ **11.** $e^z \cos z$ **12.** $\cos z^2 - \sin z$

13. Suppose f is differentiable in an open disk about zero and satisfies $f''(z) = 2f(z) + 1$. If $f(0) = 1$ and $f'(0) = i$, find the Maclaurin expansion of $f(z)$.

14. Develop the Maclaurin expansion of $1\big/\sqrt{1-z^2}$ and integrating the same show that

$$\sin^{-1}z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1.3}{2.4}\right)\frac{z^5}{5} + \left(\frac{1.3.5}{2.4.6}\right)\frac{z^7}{7} + \ldots, \quad |z| < 1.$$

15. Using $\sin z = \tan z \cos z$ and the Maclaurin series of $\sin z$ and $\cos z$, find the first four non-zero terms of the Maclaurin series of $\tan z$.

16. Find the Maclaurin's series expansion of $[(1 - z)(1 + z^2)^2]^{-1}$ and its radius of convergence.

17. Obtain the terms upto z^4 in the Maclaurin's series expansion of $f(z) = (z^2 + \sin^2 z)/(1 - \cos z)$.

18. Using the suitable series expansions prove that

(a) $\dfrac{d}{dz}(\sinh z) = \cosh z$ (b) $\dfrac{d}{dz}(\sin z) = \cos z$.

19. Using the suitable series expansions prove that

(a) $\sin\left(z + \dfrac{1}{2}\pi\right) = \cos z$ (b) $\dfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z$.

20. If the Maclaurin series of $\sec z$ is $\sec z = E_0 - \dfrac{E_2}{2!}z^2 + \dfrac{E_4}{4!}z^4 - \ldots$, then show that the numbers E_{2n}, (called the Euler's numbers), satisfy the relation

$E_0 = 1$ and ${}^{2n}C_0E_0 + {}^{2n}C_2E_2 + \ldots + {}^{2n}C_{2n}E_{2n} = 0$.

21. Using the Maclaurin series expansion of the integrand evaluate the following

(a) $erf.z = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-t^2}dt$, (error function)

(b) $Si(z) = \displaystyle\int_0^z \frac{\sin t}{t}dt$, (sine integral)

(c) $S(z) = \displaystyle\int_0^z \sin t^2\, dt$, (Fresnel integral)

16.4 LAURENT SERIES

In the preceding section we have seen that how a function $f(z)$ which is analytic at a point z_0 can be expanded as a Taylor series about z_0. Although Taylor series expansions are sufficient for many applications but sometimes it becomes necessary to expand a function $f(z)$ about a point where it is not analytic. Then we can no longer use Taylor series and require a more general form of series called Laurent series.

A Laurent series is of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad \text{...(16.14)}$$

where z_0 is a fixed point in the complex plane and the coefficients a_n may be real or complex. The first series on the right side of (16.14) containing only negative powers of $(z - z_0)$ is called the *principal part* of the Laurent series and the second series containing only positive powers of $(z - z_0)$ is called its *regular part*.

A Laurent series will converge in a domain common to the domains of convergence of the principal part and the regular part. In general, the common domain of convergence is an annulus $r < |z - z_0| < R$, where $0 < r < R$ and the sum of the Laurent series is the sum of the individual sums of the principal part and the regular part.

A simple example of the Laurent series is obtained by considering the function $f(z) = \dfrac{\cos z}{z}$ and expanding $\cos z$ as a Maclaurin series. We obtain

$$\frac{\cos z}{z} = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \ldots \qquad \text{...(16.15)}$$

The principal part of the series is $\dfrac{1}{z}$, which is convergent for all $z \neq 0$.

The regular part of the series is $-\dfrac{z}{2!} + \dfrac{z^3}{4!} - \dfrac{z^5}{6!} + \ldots$.

Using ratio test, it can be verified that it converges for all z. Hence, the annulus in which the Laurent series (16.15) converges is the complex plane except the single point at the origin.

Next, we state and prove the main result on Laurent series.

Theorem 16.4 (Laurent series): *A function $f(z)$ analytic inside the annulus $r < |z - z_0| < R$ and on the bounding circles C_1: $|z - z_0| = R$ and C_2: $|z - z_0| = r$, can be expressed as a unique Laurent series given as*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \qquad \text{...(16.16)}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z^\bullet)}{(z^\bullet - z_0)^{n+1}}\, dz^\bullet, \quad n = 0, \pm 1, \pm 2, \ldots; \qquad \text{...(16.17)}$$

and the circles C_1, C_2 and C: $|z - z_0| = \rho$, $r < \rho < R$ are positively oriented.

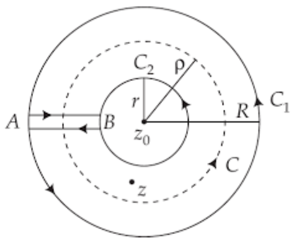


Fig. 16.2

Proof. Let the annulus $r < |z - z_0| < R$ be the one as shown in Fig. 16.2 with its center at z_0, outer boundary a circle C_1 of radius R, inner boundary a circle C_2 of radius r and C is the circle $|z - z_0| = \rho$, $r < \rho < R$, inside the annulus. Let z be any arbitrary fixed point inside the annulus $r < |z - z_0| < R$ and inside C.

Introduce a slit AB in the region enclosed by C_1 and C_2, the function $f(z)$ is analytic in the region D bounded by C_1 (anticlockwise), AB, C_2 (clockwise) and then BA, refer Fig.(16.2). Thus, if z is any point in D, we have

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_1}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet + \int_{AB}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet + \int_{C_2\text{ (clockwise)}}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet + \int_{BA}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet\right]$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet - \frac{1}{2\pi i}\int_{C_2}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet, \qquad \text{...(16.18)}$$

where both C_1 and C_2 are traversed in anticlockwise sense.

The first integral in (16.18) is precisely the same as in case of Taylor series, refer Eq. (16.6), so expanding $\frac{1}{z^\bullet - z}$ on the same lines, we get

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \qquad \text{...(16.19)}$$

where
$$a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z^\bullet)}{(z^\bullet - z_0)^{n+1}}dz^\bullet. \qquad \text{...(16.20)}$$

Using the principle of deformation of path we can replace C_1 by C in (16.20) since the point z_0 where the integrand in (16.20) is not analytic, does not lie in the annulus, and hence a_n in (16.20) is same as (16.17) for $n = 0, 1, 2, \ldots$.

For the second integral in (16.18), $z^\bullet$ lies on C_2. We write

$$\frac{1}{z^\bullet - z} = \frac{1}{(z^\bullet - z_0) - (z - z_0)} = \frac{-1}{(z - z_0)}\left[1 - \frac{z^\bullet - z_0}{z - z_0}\right]^{-1} \qquad \text{...(16.21)}$$

Since $\left|\frac{z^\bullet - z_0}{z - z_0}\right| < 1$, expanding the right side of (16.21) we obtain

$$\frac{1}{z^\bullet - z} = -\frac{1}{z - z_0}\left[1 + \frac{z^\bullet - z_0}{z - z_0} + \left(\frac{z^\bullet - z_0}{z - z_0}\right)^2 + \ldots + \left(\frac{z^\bullet - z_0}{z - z_0}\right)^n\right] - \frac{1}{z - z^\bullet}\left(\frac{z^\bullet - z_0}{z - z_0}\right)^{n+1} \qquad \text{...(16.22)}$$

Multiplying both sides of Eq. (16.22) by $-f(z^\bullet)/2\pi i$ and integrating over C_2, we get

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z^\bullet)}{z^\bullet - z}dz^\bullet = \frac{1}{2\pi i}\oint_{C_2}\frac{f(z^\bullet)}{(z - z_0)}dz^\bullet + \frac{1}{2\pi i}\oint_{C_2}\frac{(z^\bullet - z_0)f(z^\bullet)}{(z - z_0)^2}dz^\bullet + \ldots + \frac{1}{2\pi i}\oint_{C_2}\frac{(z^\bullet - z_0)^n}{(z - z_0)^{n+1}}f(z^\bullet)dz^\bullet$$

$$+ R_n(z), \qquad \text{...(16.23)}$$

where $R_n(z)$ is the remainder term given by

$$R_n(z) = \frac{1}{2\pi i}\oint_{C_2} \frac{(z^\bullet - z_0)^{n+1}}{(z - z_0)^{n+1}} \frac{f(z^\bullet)}{z - z^\bullet} dz^\bullet$$

Further since, $f(z)$ is analytic in the annulus $r < |z - z_0| < R$ and $z - z^\bullet \neq 0$ it follows that $f(z^\bullet)/(z - z^\bullet)$ is continuous and hence bounded on C_2. Thus, there exists a real positive number M, such that $|f(z)/(z - z^\bullet)| \leq M$ for all $z^\bullet$ on C_2. Hence

$$|R_n(z)| = \left|\frac{1}{2\pi i}\oint_{C_2} \frac{(z^\bullet - z_0)^{n+1}}{(z - z_0)^{n+1}} \frac{f(z^\bullet)}{z - z^\bullet} dz^\bullet\right| \leq \frac{M}{2\pi}\left|\left(\frac{r}{z - z_0}\right)^{n+1}\right| . 2\pi r \to 0 \text{ as } n \to \infty,$$

since $|r/(z - z_0)| < 1$.

Using the principle of deformation of path we can replace C_2 by C and hence as $n \to \infty$, (16.23) yields

$$-\frac{1}{2\pi i}\oint_{C_2} \frac{f(z^\bullet)}{z^\bullet - z} dz^\bullet = \frac{1}{2\pi i}\oint_C \frac{f(z^\bullet)}{(z - z_0)} dz^\bullet + \frac{1}{2\pi i}\oint_C \frac{(z^\bullet - z_0)}{(z - z_0)^2} f(z^\bullet) dz^\bullet + \ldots + \frac{1}{2\pi i}\oint_C \frac{(z^\bullet - z_0)^{n-1}}{(z - z_0)^n} f(z^\bullet)\, dz^\bullet + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}, \qquad \ldots(16.24)$$

where $a_{-n} = \frac{1}{2\pi i}\oint_C (z^\bullet - z_0)^{n-1} f(z^\bullet)\, dz^\bullet$ is (16.17) when n is replaced by $-n$.

From Eqs. (16.18), (16.19) and (16.24), we obtain (16.16), the Laurent expansion.

Remarks

1. The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique. However, $f(z)$ may have different Laurent series in two annuli with the same center.

2. Since $f(z)$ is not given to be analytic inside the closed contour C, the coefficients of the positive powers of $(z - z_0)^n$ in the Laurent series that is, $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^\bullet)}{(z^\bullet - z_0)^{n+1}} dz^\bullet$ can not be replaced by $\frac{f^{(n)}(z_0)}{n!}$. However, if $r \to 0$ and $f(z)$ is analytic at z_0 also, then $a_{-n} = 0$ and $a_n = \frac{f^{(n)}(z_0)}{n!}$ and the Laurent series of $f(z)$ reduces to Taylor series.

3. As in case of Taylor series, to obtain Laurent series whenever applicable we simply expand $f(z)$ by binomial theorem instead of finding coefficients a_n by complex integration.

Example 16.14: Find all Taylor and Laurent series expansion of $f(z) = \dfrac{1}{6 - z - z^2}$ with center 0.

Solution: The function $f(z) = \dfrac{1}{6 - z - z^2} = \dfrac{1}{(2 - z)(z + 3)}$ has singularities at the points $z = 2, -3$. So we find the expansions of $f(z)$ with center 0 in the regions

(a) $|z| < 2$, (b) $2 < |z| < 3$, and (c) $|z| > 3$.

We have, $f(z) = \dfrac{1}{6 - z - z^2} = \dfrac{1}{(2 - z)(z + 3)} = \dfrac{1}{5}\left[\dfrac{1}{2 - z} + \dfrac{1}{z + 3}\right]$

(a) For $|z| < 2$, since $|z/2| < 1$ and $|z/3| < 1$, write

$$f(z) = \frac{1}{5}\left[\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + \frac{1}{3}\left(1 + \frac{z}{3}\right)^{-1}\right] = \frac{1}{5}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\left[\frac{1}{2^{n+1}} + \frac{(-1)^n}{3^{n+1}}\right]z^n$$

This expansion contains no principal part and since $f(z)$ is analytic in $|z| < 2$, the expansion is just the Maclaurin series expension of $f(z)$ in $|z| < 2$.

(b) For $2 < |z| < 3$, since $|2/z| < 1$ and $|z/3| < 1$, write

$$f(z) = \frac{1}{5}\left[-\frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} + \frac{1}{3}\left(1 + \frac{z}{3}\right)^{-1}\right] = \frac{1}{5}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]$$

$$= \frac{1}{5}\left[-\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} + \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}\right] = \sum_{n=1}^{\infty}\left(-\frac{2^{n-1}}{5}\right)\frac{1}{z^n} + \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{5.3^{n+1}}\right)z^n$$

The first summation represents the principal part and the second summation the regular part of the Laurent series expansion of $f(z)$ in the domain $2 < |z| < 3$.

(c) For $|z| > 3$, since $|2/z| < 1$ and $|3/z| < 1$, write

$$f(z) = \frac{1}{5}\left[-\frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} + \frac{1}{z}\left(1 + \frac{3}{z}\right)^{-1}\right] = \frac{1}{5}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n\right]$$

$$= \left[\sum_{n=1}^{\infty}\left(-\frac{2^{n-1}}{5}\right) + \left(\frac{(-1)^{n-1}3^{n-1}}{5}\right)\right]\frac{1}{z^n}$$

This expansion contains only principal part.

Example 16.15: Expand $f(z) = \exp\left(z + \frac{1}{z}\right)$ as a Laurent series about the origin.

Solution: The function $f(z)$ is analytic everywhere except at the origin which is a singular point and hence $f(z)$ can be expanded about the origin in Laurent series of the form

$$\exp\left(z + \frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}, \text{ for } |z| > 0. \qquad ...(16.25)$$

To determine the coefficients $a_{\pm n}$, we write

$$f(z) = (\exp z)\left(\exp\frac{1}{z}\right) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots\right)\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \ldots\right) \qquad ...(16.26)$$

The constant term in (16.26) is

$$a_0 = 1 + 1 + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} + \frac{1}{(4!)^2} + \ldots = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}$$

Also from (16.26) we observe that the coefficients a_n and a_{-n} are equal so it is sufficient to determine a_n only. Now a_n the coefficient of z^n in (16.26) is

$$a_n = \frac{1}{n!} + \frac{1}{1!(n+1)!} + \frac{1}{2!(n+2)!} + \frac{1}{3!(n+3)!} + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}$$

Hence, Laurent expansion of $f(z)$ is (16.25), where

$$a_n = a_{-n} = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}$$

and the region of convergence is $|z| > 0$.

Example 16.16: Expand $f(z) = \frac{7z - 2}{z(z+1)(z-2)}$ as a Laurent series in the region $1 < |z+1| < 3$.

Solution: Set $z + 1 = w$, the function $f(z)$ becomes

$$f(z) = \frac{7(w-1) - 2}{w(w-1)(w-3)} = \frac{7w - 9}{w(w-1)(w-3)} = -\frac{3}{w} + \frac{1}{w-1} + \frac{2}{w-3}.$$

The region is $1 < |w| < 3$. Since $|1/w| < 1$ and $|w/3| < 1$, write

$$f(z) = -\frac{3}{w} + \frac{1}{w}\left(1 - \frac{1}{w}\right)^{-1} - \frac{2}{3}\left(1 - \frac{w}{3}\right)^{-1} = -\frac{3}{w} + \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{1}{w}\right)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{w}{3}\right)^n$$

$$= -\frac{2}{w} + \sum_{n=2}^{\infty}\left(\frac{1}{w}\right)^n - \sum_{n=0}^{\infty}\left(\frac{2}{3^{n+1}}\right)w^n$$

or, $$f(z) = -\frac{2}{1+z} + \sum_{n=2}^{\infty} \frac{1}{(1+z)^n} - \sum_{n=0}^{\infty} \frac{2}{3^{n+1}}(1+z)^n,$$

which is valid in the region $1 < |z + 1| < 3$.

Example 16.17: Show that the function $f(z) = \mathrm{Ln}[z/(z-1)]$ is analytic in the region $|z| > 1$. Obtain its Laurent series expansion about $z = 0$ valid in this region.

Solution: The function $f(z) = \mathrm{Ln}[z/(z-1)]$ is not analytic in the region where

$$\mathrm{Im}[z/(z-1)] = 0 \text{ and } \mathrm{Re}[z/(z-1)] \le 0.$$

Consider $$\frac{z}{z-1} = \frac{(x+iy)}{(x-1)+(iy)} = \frac{x(x-1)+y^2-iy}{(x-1)^2+y^2} = \frac{x(x-1)+y^2}{(x-1)^2+y^2} - i\frac{y}{(x-1)^2+y^2}$$

Now $\mathrm{Im}[z/(z-1)] = 0$ gives $y = 0$ and $\mathrm{Re}[z/(z-1)] \le 0$ gives $x(x-1) + y^2 \le 0$, or, $x(x-1) \le 0$, or $0 \le x \le 1$.

Thus $f(z)$ is not analytic in $R = \{(x, y): 0 \le x \le 1, y = 0\}$.

Hence, it is analytic in the region $|z| > 1$.

Next, for $|z| > 1$, we have $|1/z| < 1$, consider

$$\frac{1}{z} - \frac{1}{z-1} = \frac{1}{z} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} = \frac{1}{z} - \frac{1}{z}\left[\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right] = -\sum_{n=2}^{\infty}\frac{1}{z^n}.$$

Integrating both sides of this term by term we obtain

$$\mathrm{Ln}\left(\frac{z}{z-1}\right) = \sum_{n=1}^{\infty}\frac{1}{nz^n},$$

valid in the region $|z| > 1$.

EXERCISE 16.4

Find the Laurent series that converges for $0 < |z| < R$ and determine the specific region of convergence for the following functions:

1. $z^2 e^{1/z}$ **2.** $\cos z/z^4$ **3.** e^{z^2}/z^3

4. $1/(e^z - 1)$ **5.** $z^3 \cosh(1/z)$ **6.** $\sinh z/z^5$

7. Find all possible Taylor and Laurent series expansions of the function

$$f(z) = 1/[(z+1)(z+2)^2]$$

about the point $z = 1$. Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the specific region of convergence for the Problems (**8 - 11**)

8. $e^{2z}/(z-1)^3, \quad z_0 = 1$ **9.** $z^4/(z+2i)^4, \quad z_0 = -2i$

10. $\cosh z/(z+\pi i)^2, \quad z_0 = -\pi i$ **11.** $1/(1+z^2), \quad z_0 = -i.$

12. The series expansion of the functions $1/(1 - z)$ and $1/(z - 1)$ are

$$\frac{1}{1-z} = 1 + z + z^2 + ..., \qquad \text{and} \qquad \frac{1}{z-1} = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right)$$

Adding we get $(1 + z + z^2 + ...) + \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right) = 0.$

Is this result true? Justify your answer.

13. Show that $e^{\frac{u}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} C_n z^n$, $|z| > 0$, where $C_n = J_n(u) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - u\sin\theta)d\theta$ is the Bessel function of the first kind.

14. Expand (a) $\frac{1}{z^2}\int_0^z \frac{e^t - 1}{t}dt$ (b) $\frac{1}{z^3}\int_0^z \frac{\sin t}{t}dt$

in a Laurent series that converges for $|z| > 0$.

16.5 SINGULARITIES AND ZEROS

In this section we discuss singularities and zeros of complex functions. There are various types of singularities which can be classified using the Laurent series. The zeros of an analytic function can be classified using the Taylor series.

16.5.1 Singular Points of a Function

A singular point of a function f(z) is a point at which it ceases to be analytic. For example, $1/z$ is singular at $z = 0$ and $\sin z/(z - \pi)$ is singular at $z = \pi$. The points $z = 0$ and $z = \pi$ are called the *singularities* of respectively $1/z$ and $\sin z/(z - \pi)$.

Classifications of singularities: *If $z = z_0$ is a singularity of $f(z)$ such that there exists a neighbourhood of z_0 which has no singularity of $f(z)$ other than z_0, then $z = z_0$ is called an 'isolated singularity'.*

For example, $\tan z$ has isolated singularities at $z = \pm\pi/2, \pm 3\pi/2, ...,$ while $\tan(1/z)$ has a *non-isolated singularity* at 0. Since, $\tan\left(\frac{1}{z}\right) = \frac{\sin(1/z)}{\cos(1/z)}$ has singular points when $\cos(1/z) = 0$, that is, when $1/z = (2n + 1)\pi/2$, or $z = 2/[\pi(2n + 1)]$, $n = 0, \pm 1, \pm 2, ...$. Each one of these points is an isolated singularity. Since $\tan(1/z)$ is not defined at $z = 0$, thus the point $z = 0$ is also a singular point, and we observe that $\lim_{n\to\infty} \frac{2}{\pi(2n+1)} = 0$. Therefore, the neighbourhood of the point $z = 0$, that is, $|z| < \epsilon$, how so small ϵ may be, contains many singular points of $f(z)$ other than $z = 0$ and thus $z = 0$ is not an isolated singularity of $f(z) = \tan(1/z)$.

Next, a function $f(z)$ can be expanded as a Laurent's series about an isolated singularity z_0, as

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad ...(16.27)$$

By means of the terms appearing on the right side of Eq. (16.27) we classify the isolated singularities of $f(z)$ as follow.

If the coefficients of all the negative powers of $(z - z_0)$ *in (16.27) are zeros, then* $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ *and hence the singularity of* $f(z)$ *can be removed by defining* $f(z)$ *at* $z = z_0$ *in such a way that it becomes analytic at* $z = z_0$*. Such a singularity is called a '**removable singularity**'.*

For example, the Laurent's series expansion of $f(z)$ given by

$$f(z) = \frac{1 - \cos z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n-1}, \quad 0 < |z| < \infty$$

is simply a power series having the value 0 at $z = 0$. In case we define

$$g(z) = \begin{cases} f(z), & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

then $g(z)$ is differentiable at $z = 0$, (since it has power series expansion about $z = 0$); thus it has been possible to extend $f(z)$ to a function $g(z)$ which is differentiable at zero, we call zero a removable singularity of f.

To test whether the singularity $z = z_0$ is removable we simply find $\lim_{z \to z_0} f(z)$. If the limit exists and is finite, then $z = z_0$ is removable singularity.

In case the principal part of (16.27) has only finitely many terms, that is, it is of form

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_m}{(z - z_0)^m}, \ (b_m \neq 0),$$

then the singularity $z = z_0$ *of* $f(z)$ *is called a '**pole**' and m is called its '**order**'. Poles of order one are also known as 'simple poles'. A pole of order two is called a 'double pole'.*

In case the principal part of (16.27) has infinitely many terms, then the singularity $z = z_0$ *of* $f(z)$ *is called an '**essential singularity**'.*

For example, the function

$$f(z) = \frac{\sin(z)}{z^3} = \frac{1}{z^2} - \frac{1}{6} + \frac{1}{120} z^2 - \frac{1}{5040} z^4 + \ldots, z \neq 0$$

has a double pole at $z = 0$; and the function

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$$

has $z = 0$ as an essential singularity.

We observe that if $f(z)$ has a pole at $z = z_0$, then $\lim_{z \to z_0} f(z) = \infty$. In case f is analytic in the region $0 < |z - z_0| < R$, then f has a pole of order m at z_0 if, and only if $\lim_{z \to z_0} (z - z_0)^m f(z)$ exists and is non-zero.

16.5.2 Zeros of an Analytic Function

A 'zero' of an analytic function $f(z)$ is a 'z' for which $f(z) = 0$. A zero is said to be of 'order' n if, not only f but also the derivatives $f', f'', \ldots, f^{(n-1)}$ are all 0 at that z, but $f^{(n)}(z) \neq 0$. A zero of order one is called a *simple zero*; and a second order zero is called a *double zero*.

For example, the function $f(z) = \sin z$ has simple zeros at $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, and the function $f(z) = (1 + z^2)^2$ has double zeros at $z = \pm i$.

At nth order zero $z = z_0$ of $f(z)$, Taylor series is of the form

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \ldots$$

The zeros of an analytic function $f(z)$ ($\neq 0$) are isolated, that is, each of them has a neighbourhood that contains no further zeros of $f(z)$. Also in case $f(z)$ is analytic at $z = z_0$ and has a zero of nth order at $z = z_0$, then $1/f(z)$ has a pole of order n at $z = z_0$. Further, we say that $f(z)$ has an nth-order zero at infinity if $g(z)$, defined as $g(z) = f(1/z)$, has such a zero at $z = 0$. A similar result holds for poles also.

Following are a few useful results regarding the poles and zeros in terms of quotients and products of two functions. The results can be proved very easily.

1. *Let $f(z) = h(z)/g(z)$, where h and g are analytic in some open disk about z_0. Suppose $h(z_0) \neq 0$ but g has a zero of order n at z_0, then f has a pole of order n at z_0.*

For example, $f(z) = (1 + 4z^3)/\sin^6 z$ has a pole of order 6 at 0, because the numerator does not vanish at 0, and the denominator has a zero of order 6 at 0. On the same argument $f(z)$ has a pole of order 6 at $n\pi$ for any integer n.

2. *Let $f(z) = h(z)/g(z)$ and suppose h and g are analytic in some open disk about z_0. Let h have a zero of order k at z_0 and g a zero of order n at z_0, with $n > k$, then f has a pole of order $n - k$ at z_0.*

For example, $f(z) = (z - 3\pi/2)^4/\cos^7 z$. Obviously the numerator has a zero of order 4 at $3\pi/2$ and the denominator has a zero of order 7 at $3\pi/2$. Thus, $f(z)$ has a pole of order 3 at $3\pi/2$.

3. *Let f has a pole of order m at z_0 and let g has a pole of order n at z_0. Then fg has a pole of order $m + n$ at z_0.*

For example let, $f(z) = 1/(z^2 \sin z)$. Now $1/z^2$ has a double pole at $z = 0$ and $1/\sin z$ has a simple pole at $z = 0$, thus $f(z)$, which is the product of these two functions has a pole of order 3 at $z = 0$.

EXERCISE 16.5

Determine the location and type of singularities of the following functions, including those at ∞

1. $z^2 - \dfrac{1}{z^2}$ **2.** $\dfrac{1}{(z^2 + a^2)^2}$ **3.** $\dfrac{\sin^2 z}{z^2}$ **4.** $\dfrac{\sinh z}{(z - \pi i)^2}$

5. $\dfrac{z - \sin z}{z^2}$ **6.** $\dfrac{1}{\cos z - \sin z}$ **7.** $(z + 1) \sin \dfrac{1}{z - 2}$ **8.** $\dfrac{e^{2z}}{(z - 1)^4}$

Determine the location and order of the zeros of the following functions

9. $\cos^2(z/2)$ **10.** $(z^2 + 1)(e^z - 1)$ **11.** $(z^4 - z^2 - 6)^3$ **12.** $\dfrac{z^4}{\sin z}$

13. $\sin \dfrac{1}{z}$ **14 .** $\dfrac{1-\cot z}{z}$ **15.** $\cos z^3$

16. Locate and classify all singularities of $f(z) = \dfrac{(\pi - z)(z^4 - 3z^2)}{\sin^2 z}$

16.6 THE RESIDUE THEOREM

Consider the evaluation of the integral $I = \oint_C f(z)dz$ taken around a simple closed path C. If $f(z)$ is analytic everywhere on and inside C, then $I = 0$ by Cauchy's integral theorem and the problem is over. If $f(z)$ has a singularity at a point $z = z_0$ inside C but is otherwise analytic on and inside C, then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots$$

which converges in some domain $0 < |z - z_0| < R$.

Let C be a simple closed path in this domain enclosing z_0, then the Laurent's coefficients are given by Eq. (16.17) and in particular the coefficient of $\dfrac{1}{z - z_0}$ is

$$b_1 = a_{-1} = \frac{1}{2\pi i} \oint_C f(z)dz, \qquad \text{...(16.28)}$$

which implies

$$\oint_C f(z)dz = 2\pi i b_1, \qquad \text{...(16.29)}$$

where C is taken in counter-clockwise sense.

*The coefficient b_1 is called the '**residue**' of $f(z)$ at $z = z_0$ and we denote it by* $b_1 = \operatorname*{Res}_{z=z_0} f(z)$.

Generally we obtain Laurent series expansion of a function $f(z)$ without actually using the integral formulae for the coefficients, thus can find b_1 easily. The knowledge of b_1 is used to evaluate the integration of $f(z)$ around the closed curve C by using (16.29). This method of evaluating the integral I is called *the residue method of contour integration.*

Example 16.18: Evaluate by residue method the integral $\oint_C z^{-4} \sin z \, dz$; C: $|z| = 1$ taken in counter-clockwise sense.

Solution: We have,

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right) = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \ldots$$

This series is convergent for all $|z| > 0$ and $f(z)$ has a pole of third order at $z = 0$ and the residue of $f(z)$ at $z = 0$ is $b_1 = -1/3!$. Thus by (16.29)

$$\oint_C \frac{\sin z}{z^4} dz = 2\pi i b = -\frac{\pi i}{3}$$

Next, we extend the idea of evaluating contour integration to include the case when the simple closed curve encloses any finite number of singularities of the function $f(z)$. We state the following result.

Theorem 16.5 (Residue theorem): *If $f(z)$ is analytic in a closed curve C except at a finite number of singular points z_k, $k = 1, 2, \ldots$, within C, then*

$$\oint_C f(z)dz = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} f(z),$$

that is, value of this integral is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities of $f(z)$ enclosed by C.

Proof. Enclose each singularity z_k within a closed contour C_k as shown in Fig. (16.3) so that each C_k is in the interior of C and encloses exactly one singularity and does not intersect any other C_j. By the application of Cauchy's integral theorem to the multiply connected region, refer Theorem 15.3, we obtain

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z),$$

the desired result.

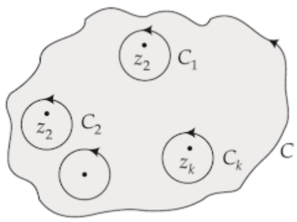


Fig. 16.3

To apply residue theorem, if we need to write the Laurent expansion of $f(z)$ about each singularity z_k to find the coefficient of $1/(z - z_k)$ term, then the theorem would have not been so advantageous to apply in many instances but, there are efficient ways of calculating residues and this makes the residue theorem quite useful.

16.6.1 Calculation of Residues

We will now develop some methods for the calculation of residues.

(a) *If $f(z)$ has a simple pole at $z = z_0$, then $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0)f(z)$.*

Since, in case of simple pole at $z = z_0$ the Laurent series expansion about z_0 is

$$f(z) = \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n,\ 0 < |z - z_0| < R.$$

This gives, $(z - z_0)f(z) = b_1 + \sum_{n=0}^{\infty} a_n(z - z_0)^{n+1}$, which implies $\lim_{z\to z_0} (z - z_0)f(z) = b_1$

Thus $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0)f(z)$ ...(16.30)

Next, we give somewhat simpler formula for the residue at a simple pole for functions of the form $f(z) = p(z)/q(z)$, both p, q analytic, $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, so that $f(z)$ has a simple pole at z_0.

Since $q(z)$ has a simple zero at z_0, the Taylor series of $q(z)$ about z_0 is

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \ldots$$

Consider, $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0)f(z) = \lim_{z \to z_0} (z - z_0)\frac{p(z)}{q(z)}$

$$= \lim_{z \to z_0} \frac{(z - z_0)p(z)}{(z - z_0)\left[q'(z_0) + \frac{(z - z_0)}{2!} q''(z_0) + \ldots\right]} = \frac{p(z_0)}{q'(z_0)}.$$

Thus, $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{p(z_0)}{q'(z_0)}$ ...(16.31)

(b) *If $f(z)$ has a pole of order m at z_0, then* $\operatorname*{Res}_{z=0} f(z) = \dfrac{1}{(m-1)!} \lim\limits_{z \to z_0} \dfrac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]$.

Since in case of pole of order m at $z = z_0$, the Laurent series expansion about z_0 is

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \ldots + \frac{b_1}{(z - z_0)} + \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

and thus

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \ldots + b_1(z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m}$$

Differentiating both sides $(m - 1)$ times and taking limit $z \to z_0$, we obtain

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] = (m - 1)!\, b_1$$

Hence, $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]$ ...(16.32)

Example 16.19: Determine the poles of function $f(z) = \dfrac{e^z}{(z+2i)^3(z+i)}$ and the residue at each pole.

Hence calculate $\oint_C f(z)dz$; C: $|z| = 2.5$, taken in counter-clockwise sense.

Solution: Since $\lim_{z \to -i} [(z+i)f(z)] = \lim_{z \to -i} \dfrac{e^z}{(z+2i)^3} = ie^{-i}$ is finite and non-zero, thus $f(z)$ has *a* simple pole at $z = -i$ and $\operatorname*{Res}_{z=-i} f(z) = ie^{-i}$.

Since $\lim_{z \to -2i} [(z+2i)^3 f(z)] = \lim_{z \to -2i} \dfrac{e^z}{z+i} = \dfrac{e^{-2i}}{-i} = ie^{-2i}$ is finite and non-zero, thus $f(z)$ has a pole of order 3 at $z = -2i$, and

$$\operatorname*{Res}_{z=-2i} f(z) = \frac{1}{2!} \lim_{z \to -2i} \frac{d^2}{dz^2}\left(\frac{e^z}{z+i}\right)$$

$$= \frac{1}{2} \lim_{z \to -2i} [e^z\{(z+i)^{-1} - 2(z+i)^{-2} + 2(z+i)^{-3}\}] = \frac{1}{2} e^{-2i}(2-i).$$

To evaluate the integral $\oint_C f(z)dz$; C: $|z| = 2.5$, we observe that $f(z)$ is analytic on and inside $|z| = 2.5$ at all points except at the poles $z = -i$, $2i$ and hence by residue theorem

$$\oint_C f(z)dz = 2\pi i\left[\operatorname*{Res}_{z=-i} f(z) + \operatorname*{Res}_{z=-2i} f(z)\right] = 2\pi i\left[ie^{-i} + \frac{1}{2}e^{-2i}(2-i)\right].$$

Example 16.20: Evaluate $\oint_C \tan z dz$; C: $|z| = 2$, taken counter-clockwise.

Solution: The poles of $f(z) = \sin z/\cos z$ are given by $\cos z = 0$, which gives $z = (2n+1)\,\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$.

Out of these only $z = \pm\pi/2$ are within the circle C: $|z| = 2$, using (16.31), we

$$\operatorname*{Res}_{z=\pi/2} f(z) = \lim_{z \to \pi/2} \frac{\sin z}{(\cos z)'} = \lim_{z \to \pi/2} \frac{\sin z}{(-\sin z)} = -1.$$

Similarly, $\operatorname*{Res}_{z=-\pi/2} f(z) = \lim_{z \to -\pi/2} \dfrac{\sin z}{(\cos z)'} = \lim_{z \to -\pi/2} \dfrac{\sin z}{(-\sin z)} = -1.$

Hence by residue theorem $\oint_C f(z)dz = 2\pi i\,[-1-1] = -4\pi i.$

Example 16.21: Evaluate $\oint_C \dfrac{\tan z}{z^2-1}dz$; C: $|z| = 3/2$, taken counter-clockwise.

Solution: Poles of $f(z) = \dfrac{\tan z}{z^2-1}$ are at $z = \pm 1$ and at $z = (2n+1)\,\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$.

Out of these only $z = \pm 1$ lie within the circle $|z| = 3/2$. Now

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1}(z-1)f(z) = \lim_{z \to 1} \frac{\tan z}{z+1} = \frac{1}{2} \tan 1;$$

and, $$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1}(z+1)f(z) = \lim_{z \to -1} \frac{\tan z}{z-1} = \frac{1}{2} \tan 1.$$

Thus, $$\oint_C \frac{\tan z}{z^2-1} dz = 2\pi i \left[\frac{1}{2} \tan 1 + \frac{1}{2} \tan 1\right] = 2\pi i \tan 1.$$

Example 16.22: Evaluate $\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz$, where C is the ellipse $9x^2 + y^2 = 9$ taken counter-clockwise.

Solution: The integrand is $f(z) = \frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}$. The first term has simple poles at $\pm 2i$ and ± 2. Out of these $\pm 2i$ lie inside the ellipse $9x^2 + y^2 = 9$ and ± 2 lie outside it. Using (16.31), we have

$$\operatorname*{Res}_{z=2i} \left[\frac{ze^{\pi z}}{z^4-16}\right] = \lim_{z=2i} \left[\frac{ze^{\pi z}}{4z^3}\right] = \lim_{z=2i} \left[\frac{e^{\pi z}}{4z^2}\right] = -\frac{1}{16}, \text{ since } e^{2\pi i} = 1.$$

$$\operatorname*{Res}_{z=-2i} \left[\frac{ze^{\pi z}}{z^4-16}\right] = \lim_{z=-2i} \left[\frac{ze^{\pi z}}{4z^3}\right] = \lim_{z=-2i} \left[\frac{e^{\pi z}}{4z^2}\right] = -\frac{1}{16}.$$

Next, considering the second term of the integrand $f(z)$, we have

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \ldots\right) = z + \pi + \frac{\pi^2}{2!z} + \frac{\pi^3}{3!z^2} + \ldots$$

Here $z = 0$ is the essential singularity with residue $\pi^2/2$.
Hence, by residue theorem

$$\oint_C f(z)dz = 2\pi i \left[-\frac{1}{16} - \frac{1}{16}\right] + 2\pi i \left[\frac{\pi^2}{2}\right] = \pi\left(\pi^2 - \frac{1}{4}\right)i.$$

EXERCISE 16.6

Find the residue of the following functions at each pole

1. $\frac{z^2+1}{z^2-2z}$ **2.** $\frac{z^2-2z}{(z+1)^2(z^2+1)}$ **3.** $\frac{\sin 2z}{z^6}$ **4.** $\frac{1}{1-e^z}$

5. $\cot \pi z$ **6.** $\frac{\sin z}{z^2(z^2+4)}$ **7.** $e^{1/z}$ **8.** $\frac{\cos z}{(z+i)^3}$

Evaluate the following integrals taken counter-clockwise

9. $\oint_C \frac{z-3}{z^2+2z+5}dz$, where C is the circle

(i) $|z| = 1$ (ii) $|z+1-i| = 2$, (iii) $|z+1+i| = 2$

10. $\oint_C \tan \pi z dz$; C: $|z| = 1$

11. $\oint_C \frac{z+1}{z^4-2z^3}dz$; C: $|z| = \frac{1}{2}$

12. $\oint_C \frac{e^{-z}}{z^2}dz$; C: $|z| = 1$

13. $\oint_C z^2 e^{\frac{1}{z}} dz$; C: $|z| = 1$

14. $\oint_C \frac{z \sec z}{(1-z)^2}dz$; C: $|z| = 3$

15. $\oint_C \frac{z \cos z}{(z-\pi/2)^3} dz$; C: $|z-1| = 1$

16. $\oint_C \frac{z \cosh \pi z}{z^4+13z^2+36}dz$; C: $|z+i| = \pi$

17. $\oint_C \frac{z-\cos(4iz)}{(1+z^2)(z^2-1)}dz$; C: $|z+i| = 1$

18. Obtain Laurent expansion for the function $f(z) = \frac{1}{z^2 \sinh z}$ and hence evaluate

$$\oint_C \frac{dz}{z^2 \sinh z}; \quad C: |z-1| = 2.$$

16.7 APPLICATIONS OF THE RESIDUE THEOREM TO THE EVALUATION OF REAL DEFINITE INTEGRALS

In this section we will illustrate how the residue theorem is used in evaluating certain classes of real definite integrals which are otherwise difficult to solve.

16.7.1 Integrals of the Type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

Here $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite over the interval of integration.

To evaluate this integral, let C be the unit circle taken counter-clockwise, then any point on this curve is $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. So we obtain

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}\left(z+\frac{1}{z}\right) \text{ and } \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

Also, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$, which gives, $d\theta = dz/iz$.

Substituting for $\cos\theta$, $\sin\theta$ and $d\theta$, the given integral takes the form

$$I = \oint_C f(z)dz; \quad C: |z| = 1,$$

where $f(z)$ is a rational function of z. By residue throrem this integral is equal to $2\pi i$ times the sum of the residues at those poles of $f(z)$ which lie within C.

Example 16.23: Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} = \frac{2\pi}{\sqrt{3}}$.

Solution: Put $z = e^{i\theta}$, we obtain $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$ and $d\theta = \dfrac{dz}{iz}$.

Thus, the given integral becomes

$$I = \oint_C \frac{1}{2 - \dfrac{z^2 - 1}{2iz}} \frac{dz}{iz} = -2\oint_C \frac{dz}{z^2 - 4iz - 1} = -2\oint_C f(z)dz; \quad C: |z| = 1, \qquad ...(16.33)$$

where $f(z) = \dfrac{1}{z^2 - 4iz - 1} = \dfrac{1}{[z - (2 + \sqrt{3})i][z - (2 - \sqrt{3})i]}$.

The function $f(z)$ has simple poles at $z = (2 \pm \sqrt{3})i$. Out of these $z_1 = (2 + \sqrt{3})i$ lies outside C and only $z_2 = (2 - \sqrt{3})i$ lies inside C. Also

$$\operatorname*{Res}_{z = z_2} f(z) = \lim_{z \to z_2} [[(z - (2 - \sqrt{3})i] f(z)] = \left[\frac{1}{z - (2 + \sqrt{3})i}\right]_{z = (2 - \sqrt{3})i} = \frac{1}{-2\sqrt{3}i}.$$

By residue theorem, $\displaystyle\oint_C f(z)dz = (2\pi i)\left(\frac{-1}{2\sqrt{3}i}\right) = -\frac{\pi}{\sqrt{3}}$, and hence from (16.33), $I = \dfrac{2\pi}{\sqrt{3}}$.

Example 16.24: Apply residue theorem to evaluate the integral

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2}\, d\theta, \quad -1 < p < 1.$$

Solution: Let $I = \displaystyle\int_0^{\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2} d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2} d\theta, \text{ since } \int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta, \text{ when } f(2a - \theta) = f(\theta).$$

Put $z = e^{i\theta}$, we obtain $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$ and $d\theta = \frac{dz}{iz}$

Thus the integral I becomes

$$I = \frac{1}{2}\oint_C \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{1 - p\left(z + \frac{1}{z}\right) + p^2}\frac{dz}{iz} = -\frac{1}{4i}\oint_C \frac{z^4 + 1}{z^2(z-p)(pz-1)}dz = -\frac{1}{4i}\oint_C f(z)\,dz\,, \quad \text{...(16.34)}$$

where $f(z) = \frac{z^4 + 1}{z^2(z-p)(pz-1)}$ and $C: |z| = 1$.

The function $f(z)$ has simple poles at $z = p$ and $\frac{1}{p}$ and a double pole at $z = 0$. Out of these 0 and p, since $|p| < 1$, are within the unit circle C. Also

$$\operatorname*{Res}_{z=p} f(z) = \lim_{z \to p}(z-p)f(z) = \left[\frac{z^4 + 1}{z^2(pz-1)}\right]_{z=p} = \frac{p^4 + 1}{p^2(p^2 - 1)}$$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z \to 0}\frac{d}{dz}[z^2 f(z)] = \lim_{z \to 0}\frac{d}{dz}\left[\frac{z^4 + 1}{(z-p)(pz-1)}\right] = \frac{d}{dz}\left[\frac{z^4 + 1}{pz^2 - (p^2+1)z + p}\right]_{z=0}$$

$$= \left[\frac{[pz^2 - (p^2+1)z + p](4z^3) - (z^4 + 1)[2pz - (p^2 + 1)]}{[pz^2 - (p^2+1)z + p]^2}\right]_{z=0} = \frac{p^2 + 1}{p^2}.$$

By residue theorem, $\oint_C f(z)dz = 2\pi i\left[\frac{p^4 + 1}{p^2(p^2 - 1)} + \frac{p^2 + 1}{p^2}\right] = -\frac{4\pi i p^2}{1 - p^2}$; hence from (16.34)

$$I = -\frac{1}{4i}\left(-\frac{4\pi i p^2}{1 - p^2}\right) = \frac{\pi p^2}{1 - p^2}.$$

Example 16.25: Prove that $\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}$, when $0 < b < a$.

Solution: Let $I = \int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$, since $\int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta$, when $f(2a - \theta) = f(\theta)$.

Put $z = e^{i\theta}$, we obtain $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$. Thus I becomes

$$I = \frac{1}{2}\oint_C \frac{dz/iz}{a + b\frac{1}{2}(z + 1/z)} = \frac{2}{i}\oint_C \frac{dz}{bz^2 + 2az + b} = \frac{2}{i}\oint_C f(z)dz, \qquad ...(16.35)$$

where $f(z) = \frac{1}{(bz^2 + 2az + b)}$ and $C: |z| = 1$.

The poles of $f(z)$ are at the roots of equation $bz^2 + 2az + b = 0$, which are

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$ and $\beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$

It is easy to verify that $|\beta| > 1$ and since $\alpha\beta = 1$, thus $|\alpha| < 1$. So $z = \alpha$ is the only simple pole inside the unit circle $|z| = 1$.

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z - \alpha)f(z) = \left.\frac{1}{b(z - \beta)}\right|_{z=\alpha} = \frac{1}{b(\alpha - \beta)} = \frac{1}{2\sqrt{a^2 - b^2}}$$

By residue theorem $\oint_C f(z)dz = 2\pi i\left(\frac{1}{2\sqrt{a^2 - b^2}}\right) = \frac{\pi i}{\sqrt{a^2 - b^2}}$; hence from (16.35)

$$I = \frac{2}{i}\left(\frac{\pi i}{\sqrt{a^2 - b^2}}\right) = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

16.7.2 Improper Integrals of the Type $\int_{-\infty}^{\infty} f(x)dx$

When $f(x)$ is a real rational function whose denominator is non-zero for all real x and is of degree at least two units higher than the degree of the numerator, then the improper integral of type under discussion converges and we can write

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

To evaluate this integral, we consider the corresponding contour integral $\oint_C f(z)dz$, where C is a closed path from $-R$ to R and then R to $-R$ along C_R, as shown in Fig. 16.4.

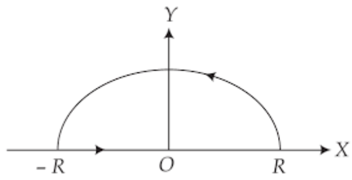


Fig. 16.4

Since $f(x)$ is rational thus $f(z)$ has finitely many poles in the upper half-plane and if we choose R large enough such that C includes all these poles, then by residue theorem, under the assumption that $f(z)$ has no singular point on the real axis, we have

$$\oint_C f(z)dz = \oint_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i\, \Sigma \text{Res}\, f(z), \quad ...(16.36)$$

where the summation is over all the residues of $f(z)$ corresponding to the poles of $f(z)$ in the upper half-plane.

Finally, taking $R \to \infty$, we obtain from (16.36)

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\, \Sigma \text{Res}\, f(z), \quad ...(16.37)$$

provided $\oint_{C_R} f(z)dz \to 0$.

To prove this, set $z = Re^{i\theta}$, then C_R is $|z| = R, 0 \le \theta \le \pi$, and under the assumption that the degree of the denominator of $f(z)$ is at least two units higher than that of the numerator, we have $|f(z)| < M/R^2$, for sufficiently large M. Thus by ML-inequality

$$\left| \int_{C_R} f(z)dz \right| < \frac{M}{R^2}\pi R = \frac{M\pi}{R} \quad ...(16.38)$$

which tends to 0 as $R \to \infty$. Thus (16.37) holds.

Example 16.26: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6 + 64}dx$.

Solution: Consider the contour integral $\displaystyle\oint_C \frac{1}{z^6 + 64}dz = \int_C f(z)dz,$

where C is the contour consisting of the semicircle C_R of radius R and the segment of the real axis from $-R$ to R as in Fig. 16.4. The integrand $f(z)$ has six simple poles at the points given by

$$z = (-64)^{1/6} = 2e^{i(\pi + 2n\pi)/6}, \quad n = 0, 1, 2, 3, 4, 5.$$

These are: $z_1 = 2e^{\pi i/6}$, $z_2 = 2e^{3\pi i/6}$, $z_3 = 2e^{5\pi i/6}$, $z_4 = 2e^{7\pi i/6}$, $z_5 = 2e^{9\pi i/6}$, $z_6 = 2e^{11\pi i/6}$.

Out of these z_1, z_2 and z_3 lies only in the upper half-plane as shown in Fig. 16.5. We have

$$\operatorname*{Res}_{z=z_1} f(z) = \operatorname*{Res}_{z=2e^{\pi i/6}} \left(\frac{1}{z^6+64}\right) = \left[\frac{1}{6z^5}\right]_{z=2e^{\pi i/6}} = \frac{1}{192} e^{-5\pi i/6},$$

Similarly, $$\operatorname*{Res}_{z=z_2} f(z) = \operatorname*{Res}_{z=2e^{3\pi i/6}} \left(\frac{1}{z^6+64}\right) = \frac{1}{192} e^{-15\pi i/6} = \frac{1}{192} e^{-3\pi i/6}$$

and

$$\operatorname*{Res}_{z=z_3} f(z) = \operatorname*{Res}_{z=2e^{5\pi i/6}} \left(\frac{1}{z^6+64}\right) = \frac{1}{192} e^{-25\pi i/6} = \frac{1}{192} e^{-\pi i/6}.$$

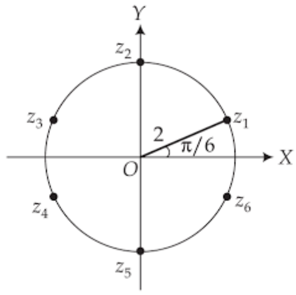


Fig. 16.5

Thus, by residue theorem

$$\int_C \frac{1}{z^6+64}\,dz = \frac{2\pi i}{192}\left[\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} + \cos\frac{3\pi}{6} - i\sin\frac{3\pi}{6} + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$$

$$= \frac{\pi i}{96}\left[\left(\cos\frac{5\pi}{6} + \cos\frac{\pi}{6}\right) - i\left(\sin\frac{5\pi}{6} + \sin\frac{\pi}{6}\right) - i\right]$$

Using $\cos\frac{5\pi}{6} + \cos\frac{\pi}{6} = 0$ and $\sin\frac{5\pi}{6} + \sin\frac{\pi}{6} = 1$, this gives

$$\oint_C \frac{1}{z^6+64}\,dz = \frac{\pi}{48} \qquad \text{...(16.39)}$$

Also, $$\oint_C \frac{1}{z^6+64}\,dz = \int_{-R}^{R} \frac{1}{x^6+64}\,dx + \int_{C_R} \frac{1}{z^6+64}\,dz \qquad \text{...(16.40)}$$

Now as $R \to \infty$, for any point on C_R, $|z| \to \infty$ and consequently the integrand in the second integral on the right side of (16.40) tends to zero and hence it vanishes and thus, from (16.39) and (16.40), we have

$$\int_{-\infty}^{\infty} \frac{1}{x^6+64}\,dx = \frac{\pi}{48}.$$

Example 16.27: Evaluate the integral $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2}$, $a > 0$.

Solution: Let $I = \displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx$

Consider the contour integral $\oint_C f(z)dz$, where $f(z) = \dfrac{z^2}{2(z^2+a^2)^2}$ and C is the closed contour consisting of the semicircle C_R of radius R and the segment of the real axis from $-R$ to R as in Fig. (16.4). The integrand $f(z)$ has poles of order 2 at $z = \pm ia$. Out of these only $z = +ia$ lies in the upper half-plane. Also

$$\operatorname*{Res}_{z=ia} f(z) = \lim_{z\to ia} \frac{d}{dz}[(z-ia)^2 f(z)] = \lim_{z\to ia} \frac{d}{dz}\left[\frac{z^2}{2(z+ia)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2z(z+ia)^2 - 2z^2(z+ia)}{(z+ia)^4}\right]_{z=ia} = -\frac{i}{8a}$$

Hence by the residue theorem

$$\oint_C f(z)dz = 2\pi i\left(-\frac{i}{8a}\right) = \frac{\pi}{4a} \qquad \text{...(16.41)}$$

Also, $$\oint_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz \qquad \text{...(16.42)}$$

When $R \to \infty$, then for any point on C_R $|z| \to \infty$ and

$f(z) = \dfrac{z^2}{2(z^2+a^2)^2} = \dfrac{1}{2z^2}\cdot\dfrac{1}{\left(1+\dfrac{a^2}{z^2}\right)^2}$ tends to 0 and consequently, the second integral on the right side of (16.42) tends to zero and hence from (16.41) and (16.42), we have

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}dx = \frac{\pi}{4a}.$$

16.7.3 Improper Real Integrals of the types $\int_{-\infty}^{\infty} f(x)\cos ax\, dx$ and $\int_{-\infty}^{\infty} f(x)\sin ax\, dx$ (Fourier Integrals).

We can express these integrals as

$$\int_{-\infty}^{\infty} f(x)\cos ax\, dx = \operatorname{Re}\int_{-\infty}^{\infty} f(x)e^{iax}dx, \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin ax\, dx = \operatorname{Im}\int_{-\infty}^{\infty} f(x)e^{iax}dx,$$

where a is real and positive.

Consider the contour integral $\oint_C f(z)e^{iaz}dz$, where C is the closed contour given by $C = C_R \cup [-R, R]$, as shown in Fig. (16.4). Using the residue theorem, we get

$$\oint_C f(z)e^{iaz}dz = 2\pi i \, \Sigma \, \text{Res}\, [f(z)e^{iaz}],$$

or, $$\int_{C_R} f(z)e^{iaz}dz + \int_{-R}^{R} f(x)e^{iax}dx = 2\pi i \, \Sigma \, \text{Res}[f(z)e^{iaz}] \qquad ...(16.43)$$

where the sum on the right side is over the residues of $f(z)e^{iaz}$ at its poles in the upper half-plane.

Consider the integral $\int_{C_R} f(z)e^{iaz}dz$, we have

$$\left|\int_{C_R} f(z)e^{iaz}\, dz\right| \le \int_{C_R} |e^{iaz}f(z)|\,|dz| = \int_{C_R} |e^{-ay}f(z)|\,|dz| \le \int_{C_R} |f(z)|\,|dz|, \qquad ...\ (16.44)$$

since $a > 0$. Now, as $R \to \infty$, if the $\int_{C_R} |f(z)|\,|dz|$ tends to zero, then from (16.44) the integral, $\int_{C_R} f(z)e^{iaz}dz$ tends to zero and hence from (16.43). We have

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \, \Sigma \, \text{Res}\, [f(z)e^{iaz}] \qquad ...(16.45)$$

Comparing the real and imaginary parts on both sides of (16.45) we get the requisite results.

Example 16.28: Evaluate the integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+\alpha^2)(x^2+\beta^2)}\, dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{(x^2+\alpha^2)(x^2+\beta^2)}\, dx$

where a, α, and β are positive numbers and $\alpha \ne \beta$.

Solution: Consider the contour integral

$$I = \oint_C \frac{e^{iaz}}{(z^2+\alpha^2)(z^2+\beta^2)}\, dz = \oint_C f(z)e^{iaz}dz,$$

where $f(z) = \dfrac{1}{(z^2+\alpha^2)(z^2+\beta^2)}$ and C is the path $C_R \cup [-R, R]$ as in Fig. (16.4).

The integrand $f(z)$ has simple poles at $z = \pm i\alpha$ and $\pm i\beta$. Out of these $i\alpha$ and $i\beta$ lie in the upper-half plane. Also

$$\operatorname*{Res}_{z=i\alpha} [e^{iaz} f(z)] = \lim_{z \to i\alpha} \left[\frac{(z - i\alpha)e^{iaz}}{(z^2 + \alpha^2)(z^2 + \beta^2)} \right] = \frac{e^{-\alpha a}}{2\alpha i(\beta^2 - \alpha^2)}$$

Similarly, $\operatorname*{Res}_{z=i\beta} [e^{iaz} f(z)] = \dfrac{e^{-\beta a}}{2\beta i(\alpha^2 - \beta^2)}$. Hence, by residue theorem

$$\oint_C e^{iaz} f(z)dz = \frac{\pi}{\beta^2 - \alpha^2} \left[\frac{e^{-\alpha a}}{\alpha} - \frac{e^{-\beta a}}{\beta} \right] \qquad \text{...(16.46)}$$

Now, $$\oint_C e^{iaz} f(z)dz = \int_{-R}^{R} e^{iax} f(x)dx + \int_{C_R} e^{iaz} f(z)dz. \qquad \text{...(16.47)}$$

We have, $\left| \int_{C_R} e^{iaz} f(z)dz \right| \le \int_{C_R} |f(z)|\,|dz|$, and here

$$f(z) = \frac{1}{(z^2 + \alpha^2)(z^2 + \beta^2)} = \frac{1}{z^4 \left(1 + \frac{\alpha^2}{z^2}\right)\left(1 + \frac{\beta^2}{z^2}\right)}$$

When $R \to \infty$, then for any point on C_R, $|z| \to \infty$ and hence $f(z) \to 0$.

Thus as $R \to \infty$, the second integral on the right side of Eq. (16.47) tends to zero and hence from (16.46) and (16.47), we obtain

$$\int_{-\infty}^{\infty} e^{iax} f(x)dz = \frac{\pi}{\beta^2 - \alpha^2} \left[\frac{e^{-\alpha a}}{\alpha} - \frac{e^{-\beta a}}{\beta} \right] \qquad \text{...(16.48)}$$

Comparing the real and imaginary parts on both sides of (16.48), we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2 + \alpha^2)(x^2 + \beta^2)} dx = \frac{\pi}{\beta^2 - \alpha^2} \left[\frac{e^{-\alpha a}}{\alpha} - \frac{e^{-\beta a}}{\beta} \right],$$

and, $$\int_{-\infty}^{\infty} \frac{\sin ax}{(x^2 + \alpha^2)(x^2 + \beta^2)} dx = 0.$$

Example 16.29: Evaluate the integral $\displaystyle\int_0^{\infty} \frac{\sin ax \sin bx}{x^2 + \alpha^2} dx$, $0 < a < b$, $\alpha > 0$.

Solution: Since, $\sin ax \sin bx = \dfrac{1}{2}[\cos(b-a)x - \cos(b+a)x]$, the given integral can be expressed as

$$I = \frac{1}{2}\int_0^{\infty} \frac{\cos(b-a)x}{x^2+\alpha^2}\,dx - \frac{1}{2}\int_0^{\infty} \frac{\cos(b+a)x}{x^2+\alpha^2}\,dx$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \frac{\cos(b-a)x}{x^2+\alpha^2}\,dx - \frac{1}{4}\int_{-\infty}^{\infty} \frac{\cos(b+a)x}{x^2+\alpha^2}\,dx$$

$$= \frac{1}{4}\operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{(b-a)ix}}{x^2+\alpha^2}\,dx - \frac{1}{4}\operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{(b+a)ix}}{x^2+\alpha^2}\,dx \qquad \text{...(16.49)}$$

We consider the contour integrals

$$I_1 = \oint_C \frac{e^{(b-a)iz}}{z^2+\alpha^2}\,dz = \oint_C e^{(b-a)iz} f(z)dz \qquad \text{...(16.50)}$$

and,

$$I_2 = \oint_C \frac{e^{(b+a)iz}}{z^2+\alpha^2}\,dz = \oint_C e^{(b+a)iz} f(z)dz, \qquad \text{...(16.51)}$$

where $f(z) = \dfrac{1}{z^2+\alpha^2}$ and C is contour given by $C_R \cup [-R, R]$ as in Fig. 16.4.

The function $f(z)$ has two simple poles at $z = \pm i\alpha$ and between these two, only $z = +i\alpha$ lies in the upper half-plane. Also

$$\operatorname*{Res}_{z=i\alpha}[e^{(b-a)iz} f(z)] = \lim_{z\to i\alpha}\left[(z-i\alpha)\frac{e^{(b-a)iz}}{z^2+\alpha^2}\right] = \frac{e^{-(b-a)\alpha}}{2\alpha i},$$

and

$$\operatorname*{Res}_{z=i\alpha}[e^{(b+a)iz} f(z)] = \lim_{z\to i\alpha}\left[(z-i\alpha)\frac{e^{(b+a)iz}}{z^2+\alpha^2}\right] = \frac{e^{-(b+a)\alpha}}{2\alpha i}$$

Hence, by residue theorem

$$I_1 = 2\pi i\left(\frac{e^{-(b-a)\alpha}}{2\alpha i}\right) = \frac{\pi}{\alpha}e^{-(b-a)\alpha} \text{ and } I_2 = 2\pi i\left(\frac{e^{-(b+a)\alpha}}{2\alpha i}\right) = \frac{\pi}{\alpha}e^{-(b+a)\alpha}.$$

Further, since $|e^{(b-a)iz}| = e^{-(b-a)y} \le 1$, $(b > a)$, and $|e^{(b+a)iz}| = e^{-(b+a)ay} \le 1$, thus we have

$$\left|\int_{C_R} \frac{e^{(b-a)iz}}{z^2+\alpha^2}dz\right| \le \int_{C_R}\left|\frac{1}{z^2+\alpha^2}\right| |dz| \qquad \text{...(16.52)}$$

and, $$\left|\int_{C_R} \frac{e^{(b+a)iz}}{z^2+\alpha^2}dz\right| \le \int_{C_R}\left|\frac{1}{z^2+\alpha^2}\right||dz|. \qquad ...(16.53)$$

When R tends to ∞, then for every point on C_R, $|z|$ tends to ∞ and hence $\dfrac{1}{z^2+\alpha^2} = \dfrac{1}{z^2}\cdot\dfrac{1}{1+\dfrac{\alpha^2}{z^2}}$ tends to 0, thus both the integrations on C_R in (16.52) and (16.53) tends to zero as $R \to \infty$. Also since

$$\oint_C e^{(b\pm a)iz} f(z)dz = \int_{-R}^{R} e^{(b\pm a)ix} f(x)dx + \int_{C_R} e^{(b\pm a)iz} f(z)dz,$$

thus from (16.50) and (16.51) we obtain respectively

$$\int_{-\infty}^{\infty} e^{(b-a)ix} f(x)dx = \frac{\pi}{\alpha}e^{-(b-a)\alpha} \text{ and } \int_{-\infty}^{\infty} e^{(b+a)ix} f(x)dx = \frac{\pi}{\alpha}e^{-(b+a)\alpha}.$$

In fact, we have

$$\text{Re}\int_{-\infty}^{\infty} \frac{e^{(b-a)ix}}{x^2+\alpha^2}dx = \frac{\pi}{\alpha}e^{-(b-a)\alpha}, \text{ and Re}\int_{-\infty}^{\infty} \frac{e^{(b+a)ix}}{x^2+\alpha^2}dx = \frac{\pi}{\alpha}e^{-(b+a)\alpha}$$

Thus from (16.49), we obtain

$$I = \frac{\pi}{4\alpha}e^{-b\alpha}[e^{a\alpha} - e^{-a\alpha}] = \frac{\pi}{2\alpha}e^{-b\alpha}\sinh(a\alpha).$$

16.7.4 Improper Integrals with Singular Points on the Real Axis

We consider an improper integral $I = \int_A^B f(x)dx$, where the integrand $f(x)$ becomes infinite at some point a in the interval of integration, that is, $\lim_{x\to a}|f(x)| = \infty$. We define the Cauchy principal value ($p_r.v.$) of the integral I as

$$p_r.v.(I) = \lim_{\epsilon\to 0}\left[\int_A^{a-\epsilon} f(x)dx + \int_{a+\epsilon}^{B} f(x)dx\right].$$

The principal value of an integral may exist, although the integral may be meaningless, refer Section 6.8 (Vol I). We shall assume here that the principal value of such an improper integral exists. Further, we shall assume $f(x)$ is rational function with degree of denominator at least two units higher than that of numerator; also the corresponding function $f(z)$ of the complex variable z is

having finite number of simple poles $a, b, c, \ldots$ on the real axis and is analytic in the upper half-plane except possibly at finite number of points, $z_1, z_2, \ldots$.

However for simplicity, in the result to be proved, we shall consider that $f(z)$ has only one simple pole $z = a$ on the real axis. The result can be generalized to finite number of poles a, b, c, etc.

Enclose this pole with semicircle C_1 with center at a and of small radius r and let C_R be the semicircle with center at the origin and radius R large enough to include all the poles of $f(z)$ in the upper half-plane.

We consider the '*indented contour*' C which is the union of the semicircle C_R, path L_1 from $-R$ to $a - r$, semicircle C_1, and then path L_2 from $a + r$ to R as per the orientations in Fig. 16.6. Thus by the residue theorem, we have

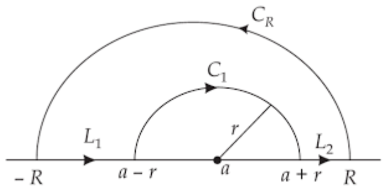


Fig. 16.6

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{L_1} f(z)dz + \int_{C_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i \sum_{z=z_k} \text{Res}\, f(z), \qquad \ldots(16.54)$$

where summation on the right is taken over all the poles, $z_1, z_2, \ldots$ of $f(z)$ in the upper-half plane.

When $r \to 0$, the semicircle C_1 contracts to a point and the line segments L_1 and L_2 expand to cover the line segment $-R$ to R on the real axis.

Further $f(z)$ has a simple pole $z = a$ on the real axis, its Laurent series expansion about $z = a$ should be of the form

$$f(z) = \frac{b_1}{z-a} + g(z); \; b_1 = \operatorname*{Res}_{z=a} f(z), \qquad \ldots(16.55)$$

where $g(z)$ is analytic on the semicircle C_1: $z = a + re^{i\theta}$, θ varing from π to 0, (note orientation of C_1 in Fig. 16.6), and for all z between C_1 and the x-axis.

Integrating (16.55) over C_1, we obtain

$$\int_{C_1} f(z)dz = b_1 \int_{\pi}^{0} \frac{ire^{i\theta}}{re^{i\theta}} d\theta + \int_{C_1} g(z)dz = -\pi i\, b_1 + \int_{C_1} g(z)dz \qquad \ldots(16.55a)$$

Now, $\left| \int_{C_1} g(z)dz \right| \le \int_{C_1} |g(z)|\,|dz| \le M\pi r$, using the ML inequality. Thus I tends to zero as $r \to 0$ and hence from (16.55a), for $r \to 0$ we have

$$\int_{C_1} f(z)dz = -\pi i \operatorname*{Res}_{z=a} f(z) \qquad \ldots(16.56)$$

Also, under the assumption that the degree of denomiator in $f(z)$ is two higher than the numerator, then as shown earlier, refer Eq. (16.38), when $R \to \infty$

$$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} |f(z)|dz \to 0. \qquad ...(16.57)$$

Thus, as $R \to \infty$ and $r \to 0$, we obtain from (16.54), (16.56) and (16.57),

$$\oint_C f(z)dz = p_r.v \int_{-\infty}^{\infty} f(x)dx - \pi i \operatorname*{Res}_{z=a} f(z) = 2\pi i \sum_{z=z_k} \text{Res } f(z)$$

or ,
$$p_r.v \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{z=z_k} \text{Res } f(z) + \pi i \operatorname*{Res}_{z=a} f(z)$$

In case of more than one simple poles on the real axis, this result is generalized to

$$p_r.v \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{z=z_k} \text{Res } f(z) + \pi i \sum_{z=a} \text{Res } f(z), \qquad ...(16.58)$$

where the first sum extends over all the poles to $f(z)$ in the upper half-plane and second extends over all the poles to $f(z)$ on the real axis.

Example 16.30: Find principal value of $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$ using the residue theorem.

Solution: Consider the function $f(z) = \dfrac{1}{(z^2-3z+2)(z^2+1)}$. It has simple poles at $z = 1, 2, i$ and $-i$.

Out of these 1 and 2 lie on real axis, i lies on the upper-half plane and $-i$ lies on the lower-half plane and is thus of no interest. We have

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} (z-1)\frac{1}{(z^2-3z+2)(z^2+1)} = \lim_{z\to 1}\frac{1}{(z-2)(z^2+1)} = -\frac{1}{2},$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2)\frac{1}{(z^2-3z+2)(z^2+1)} = \lim_{z\to 2}\frac{1}{(z-1)(z^2+1)} = \frac{1}{5},$$

and

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} (z-i)\frac{1}{(z^2-3z+2)(z^2+1)} = \lim_{z\to i}\frac{1}{(z^2-3z+2)(z+i)} = \frac{1}{6+2i} = \frac{3-i}{20}.$$

Also we note that degree of denominator in $f(z)$ is at least two more than the degree of denominator hence the line integral $\displaystyle\int_{C_R} f(z)dz$, where C_R is semicircular path of radius R in the upper half-plane, tends to zero as R tends to infinity. Thus using (16.58), we have

$$p_r.v \int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2}+\frac{1}{5}\right) = \frac{\pi}{10}.$$

Example 16.31: Show using the contour integration that

(a) $\displaystyle\int_0^{\infty} \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$ (b) $\displaystyle\int_0^{\infty} \frac{\cos mx}{x}\,dx = 0$

Solution: Since $e^{imx} = \cos mx + i \sin mx$, we consider the contour integration $\displaystyle\oint_C \frac{e^{imz}}{z}dz = \oint_C f(z)dz$, where C is the 'indented contour' given by $C = C_R \cup [-R, -r] \cup C_r \cup [r, R]$ as per the orientation shown in Fig. 16.7.

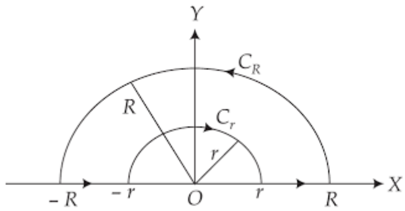


Fig. 16.7

The only singularity of $f(z) = \dfrac{e^{imz}}{z}$ is $z = 0$ which lies outside C and hence by Cauchy's integral theorem.

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz + \int_r^R f(x)dx = 0 \qquad ...(16.59)$$

Since on C_R, we have $z = Re^{i\theta}$, thus

$$\left|\int_{C_R} f(z)dz\right| = \left|\int_0^{\pi} \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} iRe^{i\theta}\,d\theta\right| \le \int_0^{\pi} |e^{imR(\cos\theta+i\sin\theta)}d\theta| \le \int_0^{\pi} e^{-mR\sin\theta}d\theta$$

$$= 2\int_0^{\pi/2} e^{-mR\sin\theta}d\theta, \qquad ...(16.60)$$

since $\displaystyle\int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta$, when $f(2a-\theta) = f(\theta)$.

For $0 \le \theta \le \pi/2$, $1 \ge \dfrac{\sin\theta}{\theta} \ge \dfrac{2}{\pi}$ and therefore, $e^{-mR\sin\theta} \le e^{-2mR\theta/\pi}$, hence from (16.60),

$$\left|\int_{C_R} f(z)dz\right| \le 2\int_0^{\pi/2} e^{-2mR\theta/\pi}d\theta = \frac{\pi}{mR}(1-e^{-mR}), \qquad ...(16.61)$$

which tends to zero as R tends to infinity.

Also as $r \to 0$, $\int_{C_r} f(z)dz = -\pi i \operatorname*{Res}_{z=0} (e^{iz}/z) = -\pi i \lim_{z \to 0} e^{iz} = -\pi i$ ...(16.62)

Hence, as $R \to \infty$ and $r \to 0$, using (16.61) and (16.62), (16.59) gives

$$p_r.v. \int_{-\infty}^{\infty} f(x)dx = i\pi \text{ or, } p_r.v. \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi.$$

Equating imaginary and real parts on both sides we get respectively

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi \text{ and } \int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0$$

or,
$$\int_{0}^{\infty} \frac{\sin mx}{x} dx = \pi/2 \text{ and } \int_{0}^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 16.32: Evaluate the integeral $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$.

Solution: We have $\frac{\sin^2 x}{x^2} = \frac{1 - \cos 2x}{2x^2} = \operatorname{Re}\left[\frac{1 - e^{2ix}}{2x^2}\right]$.

Consider the contour integral

$$\oint_C \left(\frac{1 - e^{2iz}}{2z^2}\right) dz = \oint_C f(z)dz,$$

where C is the 'indented contour' given by $C = C_R \cup [-R, -r] \cup C_r \cup [r, R]$, as shown in Fig 16.7.

The only singularity of $f(z) = \frac{1 - e^{2iz}}{2z^2}$ is at $z = 0$, a simple pole which lies outside C, and hence by Cauchy's integration theorem

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(x)dx + \int_r^R f(z)dz = 0 \qquad ...(16.63)$$

On C_R, as $R \to \infty$, we have

$$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} \left|\frac{1 - e^{2iz}}{2z^2}\right| |dz| \le \int_{C_R} \frac{1 + |e^{2iz}|}{2|z|^2} |dz| \le \frac{2}{2R^2}(\pi R) = \frac{\pi}{R} \to 0, \qquad ...(16.64)$$

since $|e^{2iz}| = e^{-2y} < 1$, for $y \ge 0$.

Also on C_r as $r \to 0$, we have

$$\int_{C_r} f(z)dz = -\pi i \operatorname*{Res}_{z=0} f(z) = -\pi i \lim_{z\to 0}\left[\frac{1-e^{2iz}}{2z}\right] = -\pi i \lim_{z\to 0}\left[\frac{-2ie^{2iz}}{2}\right] = -\pi \qquad ...(16.65)$$

Hence, as $R \to \infty$ and $r \to 0$, using (16.64) and (16.65), we obtain from (16.63)

$$p_r.v. \int_{-\infty}^{\infty} \frac{1-e^{2ix}}{2x^2}\, dx = \pi \qquad ...(16.66)$$

Comparing the real parts on both sides of (16.66), we get

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi \quad \text{or,} \int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

Example 16.33: Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2-3x+2)}\, dx$.

Solution: We have $\dfrac{\cos x}{(x^2+1)(x^2-3x+2)} = \text{Re.} \dfrac{e^{ix}}{(x^2+1)(x^2-3x+2)}$.

Consider the contour integral

$$\oint_C \frac{e^{iz}}{(z^2+1)(z^2-3z+2)}\, dz = \oint_C f(z)dz,$$

where C is the 'indented contour' given by

$$C = C_R \cup [-R, 1-r] \cup C_1 \cup [1+r, 2-r] \cup C_2 \cup [2+r, R]$$

as shown in Fig. 16.8.

Here C_1 and C_2 are both semicircles with radius r. The function $f(z) = \dfrac{e^{iz}}{(z^2+1)(z^2-3z+2)}$ has simple poles at $z = \pm i, 1, 2$. Out of these $z = i$ lies inside C, $z = 1$ and 2 lie on the real line and $z = -i$ lies in the lower-half plane. Hence, by residue theorem

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{1-r} f(z)dz + \int_{C_1} f(z)dz + \int_{1+r}^{2-r} f(z)dz + \int_{C_2} f(z)dz + \int_{2+r}^{R} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

...(16.67)

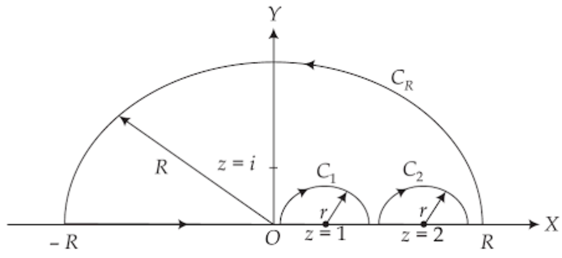


Fig. 16.8

Now, $\underset{z=i}{\text{Res}}\, f(z) = \lim_{z \to i} \left[(z-i) \dfrac{e^{iz}}{(z^2+i)(z^2-3z+2)} \right] = \dfrac{e^{-1}}{2i(1-3i)} = \dfrac{1}{2e(3+i)} = \dfrac{3-i}{20e}$. ...(16.68)

Also, $\left| \int_{C_R} f(z)dz \right| \le \int_{C_R} \left| \dfrac{e^{iz}}{(z^2+1)(z^2-3z+2)} \right| |dz| \le \dfrac{\pi R}{(R^2+1)(R^2-3R+2)} \to 0$, as $R \to \infty$, ...(16.69)

since $|e^{iz}| = e^{-y} \le 1$ for $y \ge 0$.

Also when $r \to 0$,

$$\int_{C_1} f(z)dz = -\pi i \underset{z=1}{\text{Res}}\, f(z) = -\pi i \lim_{z \to 1} (z-1) f(z)$$

$$= -\pi i \lim_{z \to 1} \frac{e^{iz}}{(z^2+1)(z-2)} = -\pi i \frac{e^{i}}{-2} = \frac{i\pi}{2} e^{i} \qquad \text{...(16.70)}$$

and, $$\int_{C_2} f(z)dz = -\pi i \underset{z=2}{\text{Res}}\, f(z) = -\pi i \lim_{z \to 2} (z-2) f(z)$$

$$= -\pi i \lim_{z \to 2} \frac{e^{iz}}{(z^2+1)(z-1)} = -\pi i \frac{e^{2i}}{5.1} = -\frac{i\pi}{5} e^{2i} \qquad \text{...(16.71)}$$

Thus, as $R \to \infty$ and $r \to 0$, using (16.68), (16.69), (16.70) and (16.71), we obtain from (16.67),

$$pr.v \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2-3x+2)} dx = \frac{\pi(1-3i)}{10e} - \frac{i\pi e^{i}}{2} + \frac{i\pi}{5} e^{2i}$$

$$= \frac{\pi(1-3i)}{10e} - \frac{i\pi}{2} [\cos(1) + i \sin(1)] + \frac{i\pi}{5} [\cos(2) + i \sin(2)]$$

Equating the real parts on each side of this equation we obtain

$$p_r.v. \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2-3x+2)}\,dx = \frac{\pi}{10}\left(\frac{1}{e} + 5\sin(1) - 2\sin(2)\right)$$

16.7.5 Solutions of a few more Improper Real Integrals using Residues

We consider the following examples:

Example 16.34: Evaluate the integral $\int_0^{\infty} \frac{x^{1/3}}{(x+1)^2}\,dx$.

Solution: Consider the contour integral $I = \oint_C \frac{z^{1/3}}{(z+1)^2}\,dz = \oint_C f(z)dz,$

where $f(z) = \frac{z^{1/3}}{(z+1)^2}$ and C is a suitably chosen contour as shown in Fig. 16.9. The outer circle is of radius R and the inner circle is of radius r. The contour consists of four parts : AB, BC, CD, DA. Let the outer circle taken counter clockwise sense be denoted by C_R and inner taken clockwise sense be denoted by C_r. We intend to make $R \to \infty$ and $r \to 0$.

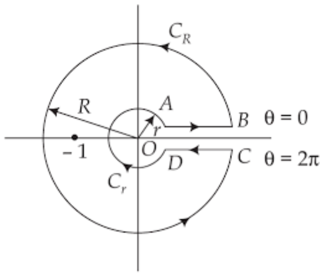


Fig. 16.9

Also in $f(z)$, the term $z^{1/3}$ is multivalued, a branch cut to the right, see Fig. 16.9, at $\theta = 0$ makes the function $f(z)$ to be single-valued and hence ensures the applicability of the residue theorem to $f(z)$.

We note that function $f(z)$ is analytic inside and on C, except for a double pole at $z = -1$, thus we have

$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{C_R} f(z)dz + \int_{C_r} f(z)dz + \int_{CD} f(z)dz = 2\pi i \operatorname*{Res}_{z=-1}[f(z)] \qquad ...(16.72)$$

Any point on C_R is $z = Re^{i\theta}$, thus

$$|f(z)| = \left|\frac{z^{1/3}}{(z+1)^2}\right| = \frac{|R^{1/3}e^{i\theta/3}|}{|z^2+2z+1|} \le \frac{R^{1/3}}{|z|^2 - 2|z| + 1} = \frac{R^{1/3}}{(R-1)^2}.$$

This gives $$\left|\int_{C_R} f(z)dz\right| \le \frac{R^{1/3}}{(R-1)^2} \cdot 2\pi R \simeq \frac{2\pi}{R^{2/3}} \to 0, \text{ as } R \to \infty. \qquad ...(16.73)$$

Similarly, $$\left|\int_{C_r} f(z)dz\right| \le \frac{r^{1/3}}{(1-r)^2} \cdot 2\pi r \simeq 2\pi r^{4/3} \to 0, \text{ as } r \to 0. \qquad \text{...(16.74)}$$

Also, $$\operatorname*{Res}_{z=-1}[f(z)] = \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)] = \lim_{z\to -1}\left(\frac{1}{3}z^{-2/3}\right)$$

$$= \frac{1}{3}(-1)^{-2/3} = \frac{1}{3}(e^{\pi i})^{-2/3} = \frac{1}{3}e^{-2\pi i/3}$$

Further on AB, we have $z = xe^{i0}$, so $z^{1/3} = x^{1/3}$, hence $f(z) = \dfrac{x^{1/3}}{(x+1)^2}$; and on CD, $z = xe^{2\pi i}$, so $z^{1/3} = x^{1/3}\, e^{2\pi i/3}$, hence $f(z) = \dfrac{x^{1/3}e^{2\pi i/3}}{(x+1)^2}$. Using these, taking $R \to \infty$, $r \to 0$ and using (16.73) and (16.74) in (16.72), we obtain

$$\int_0^\infty \frac{x^{1/3}}{(x+1)^2}dx + \int_\infty^0 \frac{x^{1/3}e^{2\pi i/3}}{(x+1)^2}dx = 2\pi i\left(\frac{e^{-2\pi i/3}}{3}\right)$$

or, $$(1 - e^{2\pi i/3})\int_0^\infty \frac{x^{1/3}}{(x+1)^2}dx = 2\pi i\left(\frac{e^{-2\pi i/3}}{3}\right)$$

or, $$\int_0^\infty \frac{x^{1/3}}{(x+1)^2}dx = \frac{2\pi i}{3}\frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}} = \frac{-2\pi i}{3}\frac{(1+i\sqrt{3})}{(3-i\sqrt{3})} = \frac{2\pi}{3\sqrt{3}}.$$

Example 16.35: Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx,\ 0 < a < 1.$

Solution: Consider the contour integral $\displaystyle\oint_C \frac{e^{az}}{e^z+1}dz = \oint_C f(z)dz$, where $f(z) = \dfrac{e^{az}}{e^z+1}$ has finite poles given by $e^z = -1 = e^{(2n+1)\pi i}$, or $z = (2n+1)\pi i$, $n = 0, \pm 1, \pm 2, \ldots$ and C is rectangular contour $ABCD$ with vertices at $A(R, 0)$, $B(R, 2\pi)$, $C(-R, 2\pi)$ and $D(-R, 0)$, as shown in Fig. 16.10.

The only pole inside the contour C is πi. Therefore by residue theorem

$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 2\pi i \operatorname*{Res}_{z=\pi i} f(z). \qquad \text{...(16.75)}$$

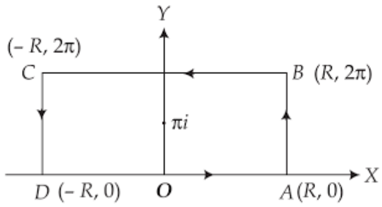


Fig. 16.10

We have, $$\operatorname*{Res}_{z=\pi i} f(z) = \operatorname*{Res}_{z=\pi i}\left[\frac{e^{az}}{e^z+1}\right] = \left[\frac{e^{az}}{(e^z+1)'}\right] = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i} \qquad ...(16.76)$$

We observe that along

$$\begin{array}{ll} AB : z = R + iy; & 0 \le y \le 2\pi; \\ BC : z = x + 2\pi i; & x \to R \text{ to } -R; \\ CD : z = -R + iy; & y \to 2\pi \text{ to } 0; \\ DA : z = x; & -R \le x \le R \end{array}$$

using these and (16.76), (16.75) becomes

$$\int_0^{2\pi} i\, f(R+iy)\, dy - \int_{-R}^{R} f(x+2\pi i)dx - \int_0^{2\pi} i\, f(-R+iy)dy + \int_{-R}^{R} f(x)dx = -2\pi i\, e^{a\pi i} \qquad ...(16.77)$$

Consider the first integral $\int_0^{2\pi} i\, f(R+iy)dy$, on the right side of (16.77) we have

$$\left|\int_0^{2\pi} i\, f(R+iy)dy\right| \le \int_0^{2\pi} |f(R+iy)|\,|dy| = \int_0^{2\pi} \left|\frac{e^{a(R+iy)}}{e^{R+iy}+1}\right| |dy| \le \frac{e^{aR}}{e^R-1}2\pi \to 0 \qquad ...(16.78)$$

as $R \to \infty$, since $a < 1$.

Similarly, consider the third integral $\int_0^{2\pi} i\, f(-R+iy)dy$, we have

$$\left|\int_0^{2\pi} i\, f(-R+iy)dy\right| \le \int_0^{2\pi} |f(-R+iy)|\,|dy| = \int_0^{2\pi} \left|\frac{e^{a(-R+iy)}}{e^{-R+iy}+1}\right| |dy| \le \frac{e^{-aR}}{e^{-R}-1}2\pi \to 0 \qquad ...(16.79)$$

as $R \to \infty$, since $a > 0$.

Thus making $R \to \infty$ and using (16.78) and (16.79), (16.77) becomes

$$-e^{2a\pi i}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i}$$

or, $$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx = \frac{2\pi i e^{a\pi i}}{e^{2a\pi i}-1} = \frac{2\pi i}{e^{a\pi i}-e^{-a\pi i}} = \frac{\pi}{\sin a\pi}$$

Example 16.36: Prove that $\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$,

Solution: Consider the contour integral $\oint_C e^{-z^2}\,dz$, where C consists of the real axis from O to A, the circular arc AB of radius R and the line $\theta = \pi/4$ from B to O as shown in Fig. 16.11.

The function $f(z) = e^{-z^2}$ is analytic everywhere inside C and hence by Cauchy's integral theorem

$$\int_{OA} e^{-z^2}\,dz + \int_{AB} e^{-z^2}\,dz + \int_{BO} e^{-z^2}\,dz = 0 \qquad ...(16.80)$$

We observe that along

$$OA : z = x\,, \qquad 0 \le x \le R$$
$$AB : z = Re^{i\theta}, \qquad 0 \le \theta \le \pi/4$$
$$BO : z = re^{i\pi/4}\,, \qquad r \to R \text{ to } 0,$$

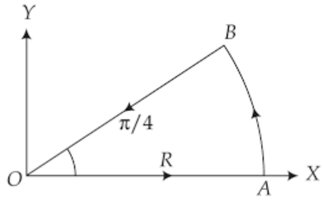


Fig. 16.11

Hence (16.80) becomes

$$\int_0^R e^{-x^2}dx + \int_0^{\pi/4} e^{-R^2e^{i2\theta}}iRe^{i\theta}d\theta - \int_0^R e^{-r^2e^{i\pi/2}}e^{i\pi/4}\,dr = 0$$

or, $$\int_0^R e^{-x^2}dx + i\int_0^{\pi/4} Re^{-R^2(\cos 2\theta + i\sin 2\theta)}e^{i\theta}d\theta - \left(\frac{1+i}{\sqrt{2}}\right)\int_0^R e^{-ix^2}dx = 0 \qquad ...(16.81)$$

As $R \to \infty$, then $\int_0^R e^{-x^2}dx \to \int_0^{\infty} e^{-x^2}dx = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$, refer (6.54)

Also $\int_0^{\pi/4} Re^{-R^2[\cos 2\theta + i\sin 2\theta]}\,e^{i\theta}d\theta \to 0$ or $R \to \infty$, since the integrend tends to zero.

Thus when $R \to \infty$, (16.81) becomes

$$\frac{\sqrt{\pi}}{2} - \frac{1+i}{\sqrt{2}}\int_0^{\infty} [\cos x^2 - i\sin x^2]dx = 0$$

or, $$\int_0^{\infty} (\cos x^2 - i\sin x^2)dx = \frac{\sqrt{\pi}}{\sqrt{2}(1+i)} = \frac{(1-i)\sqrt{\pi}}{2\sqrt{2}}$$

Equating the real and imaginary parts, we obtain

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,.$$

Example 16.37: Evaluate $\int_0^\infty e^{-x^2}\cos 2bx\,dx$,

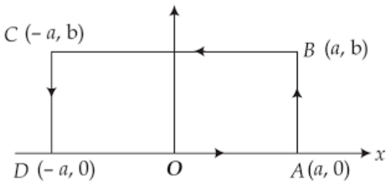


Fig. 16.12

Solution: Consider the contour integral $\oint_C e^{-z^2}dz$ where C is the rectangular contour $ABCD$ with vertices $A(a, 0)$, $B(a, b)$, $C(-a, b)$ and $D(-a, 0)$, as shown in Fig. 16.12.

The function $f(z) = e^{-z^2}$ is analytic everywhere on and inside C and hence by Cauchy's integral theorem

$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0 \qquad ...(16.82)$$

We note that along

$AB : z = a + iy;$ $\quad 0 \le y \le b$

$BC : z = x + bi;$ $\quad x \to a$ to $-a$

$CD : z = -a + iy;$ $\quad y \to b$ to 0

$DA : z = x;$ $\quad -a < x < a$

Hence, (16.82) becomes

$$\int_0^b ie^{-(a+iy)^2}dy - \int_{-a}^a e^{-(x+ib)^2}dx - \int_0^b ie^{-(-a+iy)^2}dy + \int_{-a}^a e^{-x^2}dx = 0$$

or, $$\int_{-a}^a e^{-x^2}dx - \int_{-a}^a e^{-x^2-2ibx+b^2}dx + \int_0^b ie^{-a^2-2iay+y^2}dy - \int_0^b ie^{-a^2+2iay+y^2}dy = 0.$$

or, $$\int_{-a}^a e^{-x^2}dx - e^{b^2}\int_{-a}^a e^{-x^2}.e^{-2ibx}dx + 2e^{-a^2}\int_0^b e^{y^2}\sin 2ay\,dy = 0, \qquad ...(16.83)$$

since $\sin 2ay = (e^{2iay} - e^{-2iay})/2i$.

When $a \to \infty$, $e^{-a^2} \to 0$ and $\sin 2ay$ remains between -1 and 1, hence (16.83) becomes

$$\int_{-\infty}^\infty e^{-x^2}dx - e^{b^2}\int_{-\infty}^\infty e^{-x^2}\;e^{-2ibx}dx = 0$$

or, $\int_{-\infty}^{\infty} e^{-x^2}$ $[\cos 2bx - i \sin 2bx]dx = e^{-b^2}\int_{-\infty}^{\infty} e^{-x^2}dx = 2e^{-b^2}\int_{0}^{\infty} e^{-x^2}\,dx = e^{-b^2}\Gamma(1/2) = \sqrt{\pi}\,e^{-b^2}$

Equating the real parts on both sides of this, we obtain

$$\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx\, dx = \sqrt{\pi}\, e^{-b^2}$$

Example 16.38: Determine the inverse Fourier transform of $F(w) = \dfrac{1}{w^2+1}$, defined by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{w^2+1}\, e^{iwx}\, dw.$$

Solution: Consider the contour integral

$$I = \frac{1}{2\pi}\oint_C \frac{e^{ixw}}{w^2+1}\, dw$$

in the complex w plane where C is the closed contour as shown in Fig. 16.13a, for $x > 0$ and the closed contour as shown in Fig. 16.13b, for $x < 0$.

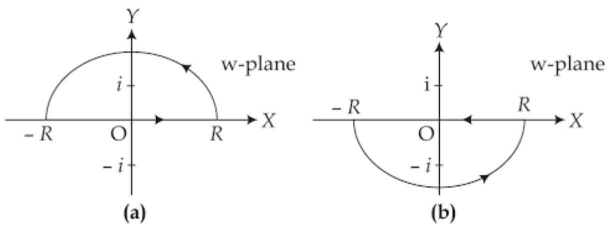


Fig. 16.13

For $x > 0$, refer Fig. 16.13a, the function $f(w) = \dfrac{1}{2\pi}\dfrac{e^{ixw}}{w^2+1}$ has singularity $w = i$ inside C, thus by residue theorem

$$\frac{1}{2\pi}\oint_C \frac{e^{ixw}}{w^2+1}\,dw = \frac{1}{2\pi}\int_{-R}^{R}\frac{e^{ixw}}{w^2+1}\,dw + \frac{1}{2\pi}\int_{C_R}\frac{e^{ixw}}{w^2+1}\,dw = 2\pi i \operatorname*{Res}_{w=i}\left[\frac{1}{2\pi}\frac{e^{ixw}}{w^2+1}\right], \qquad ...(16.84)$$

where C_R is the semicircular part of C in Fig. 16.13a. On C_R, $w = Re^{i\theta}$, thus

$$\left|\frac{1}{2\pi}\int_{C_R}\frac{e^{ixw}}{w^2+1}dw\right| \le \frac{1}{2\pi}\int_{C_R}\left|\frac{e^{ixw}}{w^2+1}\right| |dw| \le \frac{1}{2\pi}\frac{1}{R^2-1}2\pi R \to 0 \text{ as } R \to \infty.$$

Also $2\pi i \operatorname*{Res}_{w=i}\left[\frac{1}{2\pi}\frac{e^{ixw}}{w^2+1}\right] = 2\pi i.\frac{1}{2\pi}\frac{e^{ix(i)}}{2i} = \frac{e^{-x}}{2}$

Hence, as $R \to \infty$ (16.84) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixw}}{w^2+1}dw = \frac{e^{-x}}{2},\ x > 0 \qquad \text{...(16.85)}$$

Next, for $x < 0$, refer Fig (16.13b), the function $f(w) = \frac{1}{2\pi}\frac{e^{ixw}}{w^2+1}$ has singularity $w = -i$ inside C, and thus by residue theorem

$$\frac{1}{2\pi}\oint_C\frac{e^{ixw}}{w^2+1}dw = \frac{1}{2\pi}\int_R^{-R}\frac{e^{ixw}}{w^2+1}dw + \frac{1}{2\pi}\int_{C_R}\frac{e^{ixw}}{w^2+1}dw = 2\pi i \operatorname*{Res}_{w=-i}\left[\frac{1}{2\pi}\frac{e^{ixw}}{w^2+1}\right] \qquad \text{...(16.86)}$$

Also $2\pi i \operatorname*{Res}_{w=-i}\left[\frac{1}{2\pi}.\frac{e^{ixw}}{w^2+1}\right] = 2\pi i\frac{1}{2\pi}\frac{e^{ix(-i)}}{-2i} = -\frac{e^x}{2}$,

On the same lines, as above $\frac{1}{2\pi}\int_{C_R}\frac{e^{ixw}}{w^2+1}dw \to 0$ when $R \to \infty$, and hence (16.86) gives

$$\frac{1}{2\pi}\int_{\infty}^{-\infty}\frac{e^{ixw}}{w^2+1}dw = -\frac{e^x}{2}$$

or,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixw}}{w^2+1}dw = \frac{e^x}{2},\ x < 0 \qquad \text{...(16.87)}$$

Hence, from (16.85) and (16.87) we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixw}}{w^2+1}dw = \frac{e^{-|x|}}{2}$$

EXERCISE 16.7

Evaluate the following integrals

1. $\int_0^{\pi} \frac{d\theta}{k+\cos\theta}$, $(k>1)$

2. $\int_0^{\pi} \sin^4\theta\, d\theta$

3. $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}$, $|p|<1$

4. $\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta}\,d\theta$

5. $\int_0^{\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta$

6. $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$

7. $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$

8. $\int_0^{\infty} \frac{dx}{1+x^4}$

9. $\int_0^{\infty} \frac{dx}{(a^2+x^2)^2}$

10. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+4)}\,dx$

11. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3}$

12. $\int_{-\infty}^{\infty} \frac{x^{2m}}{1+x^{2n}}\,dx$; $m \neq n$ are positive integer ≤ 2.

13. $\int_0^{\infty} \frac{\cos ax}{(x^2+a^2)^2}\,dx,\ a>0$

14. $\int_{-\infty}^{\infty} \frac{\sin^2 2x}{1+x^2}\,dx$

15. $\int_0^{\infty} \frac{x^3\sin mx}{x^4+a^4}\,dx,\ a>0$

16. $\int_{-\infty}^{\infty} \frac{x^3\sin mx}{(x^2+a^2)(x^2+b^2)}\,dx$, ($m$ positive integer)

17. $\int_{-\infty}^{\infty} \frac{x}{8-x^3}\,dx$

18. $\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx$

19. $\int_{-\infty}^{\infty} \frac{\cos ax-\cos bx}{x^2}\,dx$, $(a, b \geq 0)$

20. $\int_0^{\infty} \frac{\sin^3 x}{x^3}\,dx$

21. $\int_0^{\infty} \frac{\cos ax}{1-x^4}\,dx$, $(a>0)$

22. Find a condition on α for which the integral $\int_0^{\infty} \frac{x^{\alpha-1}}{x^2+1}\,dx$ exists, and evaluate the integral subject to this condition.

23. Show that $\int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi}{2a}\ln a$, $(a>0)$

24. Evalute $\int_0^{\infty} \frac{x^{1/2}}{x^3+1}\,dx$

25. By considering integration round a suitable rectangular contour, show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \sec \frac{a}{2}, \text{ and hence } \int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x}\, dx = \frac{1}{2} \sec \frac{a}{2}, \ (-\pi < a < \pi).$$

26. Using the inversion formula and the residue theorem evaluate the inverse of the following Fourier transforms as per the definition given in Example (16.38),

(a) $\dfrac{1}{(1+w^2)^2}$ (b) $\dfrac{1}{(1+iw)^2}$

27. By considering the contour integral $\oint_C \exp(iz^2)dz$ around a suitable contour C, show that

$$\int_0^{\infty} \cos x^2 dx = \int_0^{\infty} \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

ANSWERS

Exercise 16.1 (p. 93)

1. Convergent 2. Divergent 3. Converges $p > 1$, diverges $0 < p \le 1$.
4. Converges 5. Converges 6. Divergent

Exercise 16.2 (p. 98)

1. $-1, 1$ 2. $-i\sqrt{2}, 1$ 3. $0, 0$ 4. $-2+i, \sqrt{2}$
5. $0, e^{\pi}$ 6. $0, 1$ 7. $-i, 1/e$ 8. $0, 1$
9. $0, 1/\sqrt{2}$ 10. $0, 2$ 11. $0, 1$ 12. $0, 1/2$

Exercise 16.3 (p. 104)

1. $\frac{1}{2} - \frac{1}{4}(z-2) + \frac{1}{8}(z-2)^2 - \frac{1}{16}(z-2)^3 + \ldots;\ R = 2.$

2. $e^a\left[1 + (z-a) + \frac{1}{2!}(z-a)^2 + \frac{1}{3!}(z-a)^3 + \ldots\right];\ R = \infty$

3. $1 - \frac{1}{2!}\left(z - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(z - \frac{\pi}{2}\right)^4 - \ldots.;\ R = \infty$

4. $i\left[1 + \frac{1}{2!}\left(z - \frac{\pi i}{2}\right)^2 + \frac{1}{4!}\left(z - \frac{\pi i}{2}\right)^4 + \ldots\right];\ R = \infty$

5. $-\left[1+\frac{1}{2!}(z-\pi i)^2+\frac{1}{4!}(z-\pi i)^2+\ldots\right]$; $R=\infty$

6. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}i^n(z-i)^n}{n}$; $R=1$

7. $\sum_{n=0}^{\infty}(-1)^n z^{2n}$; $R=1$

8. $z-\frac{z^3}{3}+\frac{z^5}{5}-\ldots$; $R=1$

9. $z^2-\frac{1}{3}z^4+\frac{2}{45}z^6-\frac{1}{315}z^8+\ldots$; $R=\infty$

10. $2+z+2z^2+z^3+2z^4+\ldots$; $R=1$

11. $\sum_{n=0}^{\infty}\frac{(1+i)^n+(1-i)^n}{2n!}z^n$; $R=\infty$

12. $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{4n}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$; $R=\infty$

13. $1+iz+\sum_{n=1}^{\infty}\left(\frac{2^n+2^{n-1}}{(2n)!}z^{2n}+i\frac{2^n}{(2n+1)!}z^{2n+1}\right)$; $R=\infty$

15. $1+\frac{1}{3}z^3+\frac{2}{15}z^5+\frac{17}{315}z^7+\ldots$; $R=\pi/2$

16. $1+z-z^2-z^3+2z^4+2z^5-\ldots$; $R=1$

17. $4-\frac{z^2}{3}+\frac{1}{20}z^4+\ldots$; $R=2\pi$

21. (a) $\left(2/\sqrt{\pi}\right)\left(z-z^3/3+z^5/2!\,5-z^7/3!\,7+\ldots\right)$; $R=\infty$

(b) $z-z^3/3!\,3+z^5/5!5-z^7/7!\,7+\ldots$; $R=\infty$

(c) $z^3/1!\,3-z^7/3!\,7+z^{11}/5!\,11-\ldots$; $R=\infty$

Exercise 16.4 (p. 111)

1. $\frac{1}{2}+z+z^2+\sum_{n=1}^{\infty}\frac{1}{(n+2)!z^n}$; $|z|>0$

2. $\frac{1}{z^4}-\frac{1}{2z^2}+\frac{1}{24}-\frac{1}{720}z^2+\ldots$; $|z|>0$

3. $\frac{1}{z^3}+\frac{1}{z}+\frac{1}{2}z+\frac{1}{6}z^3+\ldots$; $|z|>0$

4. $\frac{1}{z}-\frac{1}{2}+\frac{z}{12}$; $0<|z|<2\pi$

5. $z^3+\frac{1}{2}z+\frac{1}{24z}+\frac{1}{720z^3}+\ldots$; $|z|>0$

6. $\frac{1}{z^4}+\frac{1}{3!}\frac{1}{z^2}+\frac{1}{5!}+\frac{1}{7!}z^2+\ldots$; $|z|>0$

7. (i) $\sum_{n=0}^{\infty} (-1)^n [2^{-n-1} - (n+4)3^{-n-2}] (z-1)^n$, $|z-1| < 2$

(ii) $\sum_{n=0}^{\infty} (-1)^n \dfrac{2^n}{(z-1)^{n+1}} - \dfrac{1}{9}\sum_{n=0}^{\infty}(-1)^n (n+4)\left(\dfrac{z-1}{3}\right)^n$, $2 < |z-1| < 3$

(iii) $\sum_{n=0}^{\infty} (-1)^n \left[\dfrac{2^n - 3^n}{(z-1)^{n+1}} - \dfrac{3^n(n+1)}{(z-1)^{n+2}}\right]$, $|z-1| > 3$

8. $e^2\left[\dfrac{1}{(z-1)^3} + \dfrac{2}{(z-1)^2} + \dfrac{2}{(z-1)} + \dfrac{4}{3} + \dfrac{2}{3}(z-1) + \ldots\right]$; $|z-1| > 0$

9. $1 - \dfrac{8i}{(z+2i)} - \dfrac{24}{(z+2i)^2} + \dfrac{32i}{(z+2i)^3} + \ldots$; $|z+2i| > 0$

10. $\dfrac{1}{-(z+\pi i)^2} - \dfrac{1}{2} - \dfrac{1}{24}(z+\pi i)^2 + \ldots$; $|z+\pi i| > 0$

11. $\dfrac{i}{2}\dfrac{1}{z+i} + \dfrac{i}{2}\sum_{n=0}^{\infty}\dfrac{1}{(2i)^{n+1}}(z+i)^n$; $0 < |z+i| < 2$

12. No; the first is valid for $|z| < 1$ and the second is valid for $|z| > 1$; there are no common points between the two regions.

Exercise 16.5 (p. 114)

1. 0, ∞ (second order poles)
2. $\pm ia$, second order poles
3. ∞, essential singularity
4. πi, simple pole; ∞ (essential singularity)
5. 0, removable singularity
6. $\pi/4$, simple pole
7. 2, essential singularity
8. 1, pole of order 4
9. $(2n+1)\pi$, $n = 0, \pm 1, \pm 3$ (second order)
10. $\pm i$, 0, $\pm 2\pi i$, $\pm 4\pi i$; ... simple
11. $\pm\sqrt{3}$ $\pm i\sqrt{2}$; third order
12. 0, third order
13. $\pm\dfrac{1}{n\pi}$, $n = 1, 2, 3, \ldots$
14. $\dfrac{\pi}{4} + n\pi$, $n = 0, \pm 1, \pm 2 \ldots$ simple
15. $z = [(2k+1)\pi/2]^{1/3}$ and $z = \dfrac{1}{2}\sqrt{(2k+1)\pi/2}\left(1 \pm i\sqrt{3}\right)$, $k = 0, \pm 1, \pm 2, \ldots$ simple
16. First order pole at π, second order poles at $z = n\pi$, $n = -1, \pm 2, \pm 3, \ldots$

Exercise 16.6 (p. 119)

1. $-\frac{1}{2}$ (at 0), $\frac{5}{2}$ (at 2)
2. $\frac{1}{2}$ (at -1), $\frac{1}{4}(1+2i)$ (at i), $\frac{1}{4}(1-2i)$ (at $-i$)
3. $4/15$ (at 0),
4. -1, (at $\pm 2n\pi i$)
5. $1/\pi$ (at $0, \pm 1, \ldots$)
6. $\frac{1}{4}$ (at + 0), $\frac{i}{16}\sin(2i)$, (at $2i$),
7. 1, (at 0)
8. $-\frac{1}{2}\cos i$, (at $-i$)
9. (i) 0, (ii) $\pi(i-2)$, (iii) $\pi(2+i)$
10. $-4i$
11. $-\frac{3}{4}\pi i$
12. $-2\pi i$
13. $8\pi i/3e^2$
14. $2\pi i \sec 1\,(1+\tan 1)$
15. $-2\pi i$
16. $4\pi i/5$
17. $\frac{\pi}{2}(-i-\cos 4)$
18. $\frac{1}{z^3}-\frac{1}{6z}+\frac{7z}{360}-\frac{31z^6}{15120}+\ldots; -\frac{1}{3}\pi i$

Exercise 16.7 (p. 144)

1. $\frac{\pi}{\sqrt{k^2-1}}$
2. $\frac{3\pi}{8}$
3. $\frac{2\pi}{1-p^2}$
4. 0
5. $\frac{3\pi}{16}$
6. 2π
7. $\pi/3$
8. $\frac{\pi}{2\sqrt{2}}$
9. $\frac{\pi}{4a^3}$
10. $\pi/18$
11. $\frac{3\pi}{8a^5}$
12. $\frac{\pi}{[n\sin\{(2m+1)\pi/2n\}]}$
13. $\frac{\pi(1+a)}{4a^3e^a}$
14. $\frac{\pi}{2}(1-e^{-4})$
15. $\frac{\pi}{2}\, e^{-ma/\sqrt{2}}\cos\left(ma\sqrt{2}\right)$
16. $\frac{\pi[a^2e^{-ma}-b^2e^{-mb}]}{a^2-b^2}$
17. $-\sqrt{3}\pi/6$
18. π
19. $\pi(b-a)/2$
20. $3\pi/8$
21. $\frac{\pi}{4}[e^{-a}+\sin a]$
22. $\frac{\pi\cos(\alpha\pi/2)}{\sin\alpha\pi}, 0<\alpha<2, \alpha\neq 1$
24. $\pi/3$

17 Fourier Series

CHAPTER

"Fourier series arise naturally while analyzing many physical phenomena like electrical oscillations, vibrating mechanical systems, longitudinal oscillations in crystals, etc. Many functions including some discontinuous periodic functions of practical interest, which do not find Taylor series representation, can be expanded in a Fourier series and, as such, Fourier series are more universal than Taylor series. They are very powerful tools in solving certain ordinary and partial differential equations. Modern-day applications of Fourier series include areas like data compression, filtering and signal analysis, CAT scans, sattelite communication, etc."

17.1 THE FOURIER SERIES OF A FUNCTION

A Fourier series is a representation of a periodic function as a series of cosine and/or sine terms. Before studying Fourier series we need to consider periodic functions.

17.1.1 Periodic Functions

A function $f(x)$ is called a 'periodic function', if there is some positive number T such that for every x in the domain of f

$$f(x + T) = f(x); \qquad ...(17.1)$$

and the number $T > 0$, with this property, is called a 'period' of $f(x)$.

For example, $\sin x$ is periodic with period 2π, since $\sin (x + 2\pi) = \sin x$ for all x. The function $\tan x$ is periodic with period π, for $\tan (x + \pi) = \tan x$ for all x.

Examples of *non-periodic* functions are x, e^x, etc.

Further, we note that if f is periodic with period T, it is necessarily periodic with period $2T$, $3T$, $4T$ ... as well, since $f(x + 2T) = f((x + T) + T) = f(x +T) = f(x)$. Of all these possible periods, the smallest one, if it exists, is called the *'fundamental period'* of $f(x)$. For example, for $\cos x$ and $\sin x$ the fundamental period is 2π while for $\cos 2x$ and $\sin 2x$ it is π.

The function $f(x)$ = constant is periodic and every $T > 0$ is a period. Thus there is no smallest period so, $f(x)$ = constant does not have a fundamental period.

Further, if $f(x)$ and $g(x)$ are periodic with fundamental period T, then the function $h(x) = af(x) + bg(x)$, a, b being constants is also periodic with period T.

Now we define the Fourier series representation of a function $f(x)$.

17.1.2 Fourier Series

If a function $f(x)$ is periodic with period 2π and is integrable over $-\pi < x < \pi$, then the Fourier series representation of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad ...(17.2)$$

where the coefficients $a_0, a_1, a_2, ...; b_0, b_1, b_2, ...,$ called the Fourier coefficients, are determined by the function $f(x)$.

To determine the coefficients we need the following results which follow from the orthogonality property of the trigonometric functions $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$ ... $\cos nx$, $\sin nx$. However, the results can be proved otherwise also, for all integral values of m and n.

1. $$\int_{-\pi}^{\pi} \sin nx\,dx = \int_{-\pi}^{\pi} \cos nx\,dx = 0 \qquad ...(17.3)$$

2. $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases} \qquad ...(17.4)$$

3. $$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases} \qquad ...(17.5)$$

4. $$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0, \text{ for all } m \text{ and } n. \qquad ...(17.6)$$

In fact, the results from (17.3) to (17.6) hold for the interval of integration $(\alpha, \alpha + 2\pi)$, for an arbitrary α. In particular, for $\alpha = -\pi$, the interval becomes $(-\pi, \pi)$ and for $\alpha = 0$ the interval becomes $(0, 2\pi)$.

Next, we determine the coefficients a_0, a_n and b_n.

Determination of the constant term a_0. Integrating both sides of (17.2) from $-\pi$ to π and assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right). \qquad ...(17.7)$$

Using (17.3), and integrating the first term on the right side of (17.7), we obtain

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Determination of the coefficients a_n. Multiplying (17.2) by cos mx for any fixed positive integer m and integrating on both sides from $-\pi$ to π; assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\right). \quad ...(17.8)$$

Using (17.3), (17.5) and (17.6) on the right side of (17.8), we obtain

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx.$$

Determination of the coefficients b_n. Similarly multiplying (17.2) by sin mx, for any fixed positive integer m, and integrating on both sides from $-\pi$ to π; assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)\sin mx = a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,dx\right). \quad ...(17.9)$$

Using (17.3), ((17.4) and (17.6) on the right side of (17.9), we obtain

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

The results:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad ...(17.10)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad ...(17.11)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad ...(17.12)$$

for $n = 1, 2, ...$ are called the *Euler's formulae* for the Fourier coefficients a_n and b_n associated with the Fourier series representation of $f(x)$ given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad ...(17.13)$$

17.1.3 Convergence and Sum of a Fourier Series

Suppose that $f(x)$ is any given periodic function of period 2π which is continuous or merely piecewise continuous over the interval of integration. Then we can compute the Fourier coefficients a_0, a_n, b_n of $f(x)$ and use them to form the Fourier series (17.13) of $f(x)$. We would expect that the series thus obtained converges to $f(x)$ over the domain of definition of f.

Various results are available that give sufficient conditions on f for the Fourier series of $f(x)$ to represent $f(x)$. However, one such result which covers the majority of periodic functions appearing in practical applications, is stated as follows.

Theorem 17.1: *If $f(x)$ is a periodic function with period 2π and if $f(x)$ and $f'(x)$ both are piecewise continuous in the interval $-\pi \le x \le \pi$, then the Fourier series of $f(x)$ is convergent. It converges to $f(x)$ at every point x at which $f(x)$ is continuous, and to the mean value $[f(x+) + f(x-)]/2$ at every point x at which $f(x)$ is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.*

But we must note that the jumps at the points of discontinuity must be finite. Figure 17.1 shows the graph of a typical piecewise continuous function.

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x/2, & 1 < x \le 2 \\ 2x & 2 < x \le 3 \end{cases}.$$

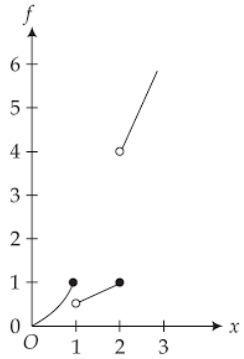


Fig. 17.1

At points of discontinuity (e.g. at $x = 1, 2$), the function $f(x)$ has 'finite' jumps.

An example of a simple function which is not piecewise continuous is

$$f(x) = \begin{cases} 0, & x = 0 \\ 1/x, & 0 < x \le 1 \end{cases}$$

since $\lim_{x \to 0^+} f(x) = \infty$ and so the jump at the discontinuity $x = 0$ is not finite and thus $f(x)$ is not piecewise continuous on [0,1].

Example 17.1 (Saw-tooth wave): Find the Fourier series for the function $f(x) = x$, $-\pi < x < \pi$, when $f(x) = f(x + 2\pi)$.

Solution: The graph of the function $f(x)$ is shown in Fig. 17.2. It is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = 0$$

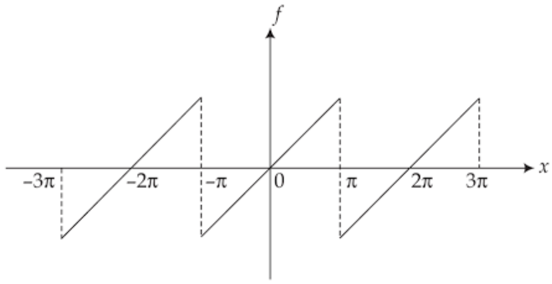


Fig. 17.2

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[-\left(\frac{x\cos nx}{n}\right)_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = -\frac{2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n}.$$

Hence Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \ldots$$

We note that the Fourier series converges to $f(x)$ at every point at which it is continuous. The function $f(x)$ has points of finite discontinuity at $x = -3\pi, -\pi, \pi, 3\pi$ ... The average of the extremes at each discontinuity is $\frac{1}{2}(\pi + (-\pi)) = 0$; and it can be verified by direct substitution that the above series converges to zero at $x = -3\pi, -\pi, \pi, 3\pi$, etc.

Example 17.2 (Rectangular wave): Find the Fourier series for the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ 4, & 0 < x < \pi \end{cases}$$

Solution: The graph of the periodic function $f(x)$ is shown in Fig. 17.3

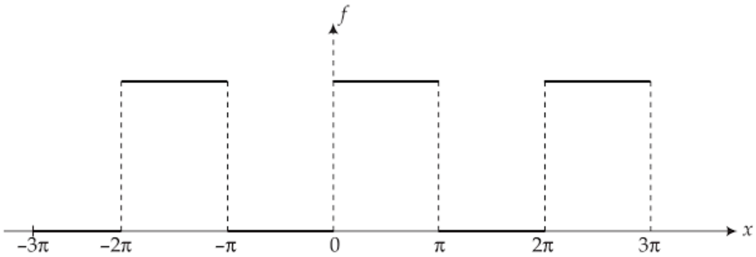


Fig. 17.3

It is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 4dx\right] = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} 4\cos nx\,dx\right] = \frac{4}{n\pi}[\sin nx]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} 4\sin nx\,dx\right]$$

$$= \frac{-4}{n\pi}[\cos nx]_0^{\pi} = \frac{4}{n\pi}[1-(-1)^n]$$

Thus, $$b_n = \begin{cases} \dfrac{8}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = 2 + \frac{8}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right) = 2 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}. \qquad \ldots(17.14)$$

We note that at points of finite discontinuity of $f(x)$, that is, at $x = -2\pi, -\pi, 0, \pi, 2\pi$, etc., the series converges to 2, the mean value $\frac{1}{2}(0+4)$ and for all others x in the domain of definition of f, the series converges to $f(x)$.

To illustrate how this convergence to f is achieved, we plot some of the partial sums of the series (17.14). The partial sums are

$$S_1(x) = 2, \quad S_2(x) = 2 + \frac{8}{\pi}\sin x, \quad S_3(x) = 2 + \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x,$$

$$S_4(x) = 2 + \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x + \frac{8}{5\pi}\sin 5x$$

and so on.

Their graphs in Fig. 17.4 show that the series is convergent and has the sum $f(x)$ Also we note that at $x = -\pi, 0, \pi$, the points of discontinuity of $f(x)$, all partial sums have the value '2', the average of the values 0 and 4 of $f(x)$.

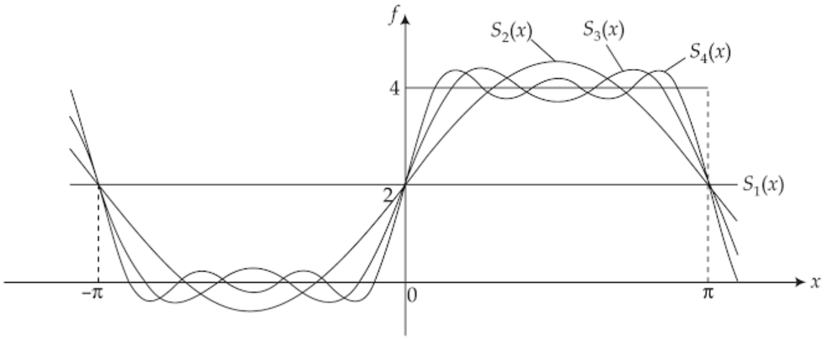


Fig. 17.4

The graph in Fig. 17.4 shows upto the first four partial sum approximations to the function $f(x)$ defined in Example (17.2). The Nth partial sum is given by

$$S_N(x) = a_0 + \sum_{n=1}^{N-1} (a_n \cos nx + b_n \sin nx)$$

and,
$$f(x) = \lim_{N\to\infty} S_N(x).$$

Remark: It should be noted that not every function has a Fourier expansion involving an infinite number of terms. For example, $f(x) = 1 + 2\sin x\cos x$ when rewritten as $f(x) = 1 + \sin 2x$ is in fact its own Fourier series.

Example 17.3 (Rectangular pulse): Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

Solution: The graph of the function $f(x)$ on the interval $(-\pi, \pi)$ is shown in Fig. 17.5.

The function is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\,dx$$

$$= \frac{1}{\pi}\left(\frac{\sin nx}{n}\right)_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

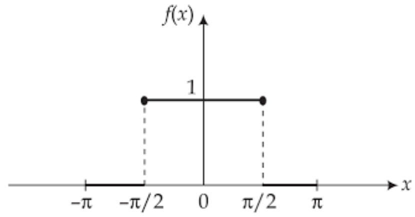


Fig. 17.5

which gives,
$$a_n = \begin{cases} \frac{2}{n\pi}(-1)^{\frac{n-1}{2}} & \text{for odd } n \\ 0, & \text{for even } n \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\,dx = \frac{1}{n\pi}[-\cos nx]_{-\pi/2}^{\pi/2} = 0.$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \ldots\right]. \quad \text{...(17.15)}$$

Figs. 17.6a and 17.6b show respectively that graphs of the first five and the first ten terms of the Fourier series expansion (17.15) in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. It can be observed that the graphs of $S_5(x)$ and $S_{10}(x)$ exhibit over and undershoots close to the discontinuities $x = -\pi/2, \pi/2$. This oscillatory behaviour of the partial sums $S_N(x)$ near a point of jump discontinuity continues even for large value N and is called the *Gibbs phenomenon*.

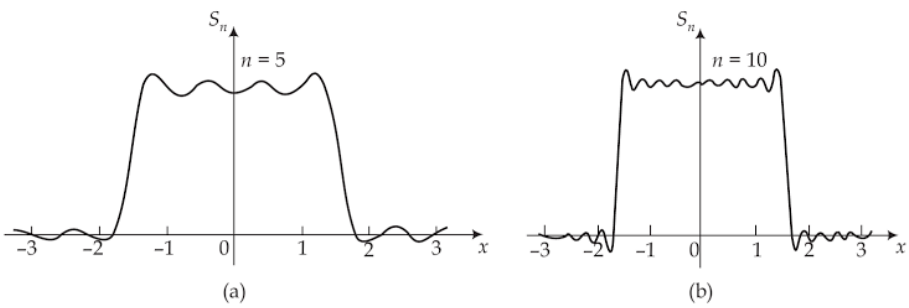


Fig. 17.6

Example 17.4: Find the Fourier series for the function $f(x) = e^{-x}$, $0 < x < 2\pi$ with $f(x + 2\pi) = f(x)$.

Solution: The function $f(x)$ is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx = -\frac{1}{2\pi}(e^{-x})_0^{2\pi} = \frac{1-e^{-2\pi}}{2\pi},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x}(-\cos nx + n\sin nx)]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x}(-\sin nx - n\cos nx)]_0^{2\pi} = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}.$$

Hence the Fourier series of $f(x)$ on $(0, 2\pi)$ is

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+1} + \frac{n\sin nx}{n^2+1}\right)\right\}.$$

17.2 FOURIER SERIES OF FUNCTIONS OF PERIOD $T = 2l$

So far we have considered the Fourier series expansion of functions with period 2π. In many applications, we need to find the Fourier series expansion of periodic functions with arbitrary period, say $2l$. The transition from period $T = 2l$ to period $T = 2\pi$ is quite simple and involves only a proportional change of scale.

Consider the periodic function $f(x)$ with period $2l$ defined in $(-l, l)$. To change the problem to period 2π, set

$$v = \frac{\pi x}{l}, \text{ which gives, } \quad x = \frac{lv}{\pi}.$$

Thus $x = \pm l$ corresponds to $v = \pm\pi$ and the function $f(x)$ of period $2l$ in $(-l, l)$ may be regarded as function $g(v)$ of period 2π in $(-\pi, \pi)$. Hence,

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nv + b_n\sin nv) \qquad \text{...(17.16)}$$

with coefficients

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)dv \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\,dv \end{aligned}\right\} \qquad \text{...(17.17)}$$

Making the inverse substitutions, $v = \dfrac{\pi x}{l}$ and $g(v) = f(x)$ in (17.16) and (17.17), we obtain the Fourier series expansion of $f(x)$ in the interval $(-l, l)$ given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

with coefficients

$$\left.\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(x)dx \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \end{aligned}\right\} \qquad \text{...(17.18)}$$

We may replace the interval of integration by any interval of length $T = 2l$, say by the interval $(0, 2l)$.

Example 17.5: Find the Fourier series for the function

$$f(x) = \begin{cases} x, & -1 < x \le 0 \\ x + 2, & 0 < x < 1, \end{cases}$$

where $f(x) = f(x + 2)$. From the series obtained deduce the sum of the series $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dots$.

Solution: The graph of the periodic function $f(x)$ in the interval $(-1, 1)$ is shown in Fig. 17.7.

The function is periodic with period 2. The Fourier coefficients are

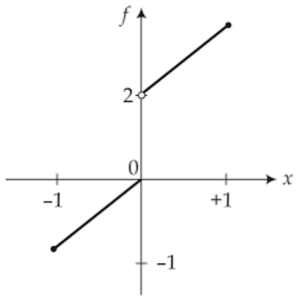


Fig. 17.7

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[\int_{-1}^{0} x\,dx + \int_{0}^{1}(x+2)dx\right] = 1,$$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1}(x+2)\cos n\pi x\,dx$$

$$= 2\int_{0}^{1}\cos n\pi x\,dx = \frac{2}{n\pi}(\sin n\pi x)_0^1 = \frac{2}{n\pi}(\sin n\pi) = 0,$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1}(x+2)\sin n\pi x\,dx$$

$$= 2\int_{0}^{1} x\sin n\pi x\,dx + 2\int_{0}^{1}\sin n\pi x\,dx = 2\left[-\frac{x\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 - 2\left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{2\cos n\pi}{n\pi} - \frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi} = \frac{2}{n\pi} - \frac{4}{n\pi}(-1)^n = \frac{2}{n\pi}[1-(-1)^n 2]$$

Thus $$b_n = \begin{cases} \dfrac{6}{n\pi}, & \text{for odd } n \\ \dfrac{-2}{n\pi}, & \text{for even } n \end{cases}$$

Hence the Fourier series of $f(x)$ on $(-1, 1)$ is

$$f(x) = 1 + \frac{2}{\pi}\left[3\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}(3\sin 3\pi x) - \frac{1}{4}\sin 4\pi x + \frac{1}{5}(3\sin 5\pi x) - \frac{1}{6}\sin 6\pi x + - \ldots\right].$$

Further, for $x = 1/2$, $f(x) = x + 2 = 1/2 + 2 = 5/2$.

Setting $x = 1/2$ on both sides of the series above, we obtain

$$\frac{5}{2} = 1 + \frac{2}{\pi}\left[3\sin\frac{\pi}{2} - \frac{1}{2}\sin\pi + \sin\frac{3\pi}{2} - \frac{1}{4}\sin 2\pi + \frac{3}{5}\sin\frac{5\pi}{2} - \frac{1}{6}\sin 3\pi + \frac{3}{7}\sin\frac{7\pi}{2} - \ldots\right]$$

$$= 1 + \frac{2}{\pi}\left[3 - 1 + \frac{3}{5} - \frac{3}{7} + \ldots\right]$$

or, $$\frac{3\pi}{4} = 3 - 1 + \frac{3}{5} - \frac{3}{7} + \ldots,$$

This gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

Example 17.6 (Half-wave rectifires): Find the Fourier series of the periodic function

$$f(x) = \begin{cases} 0, & -l < x \le 0 \\ E \sin wx, & 0 < x < l \end{cases}$$

with period $T = 2l = \dfrac{2\pi}{w}$.

Solution: The graph of the periodic function $f(x)$ with period $2\pi/w$ is shown in Fig. 17.8.

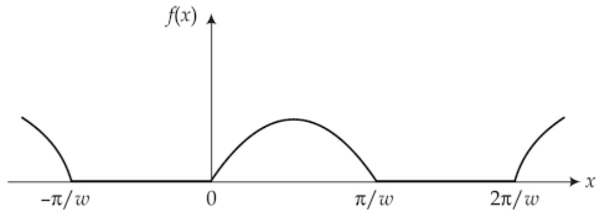


Fig. 17.8

The Fourier coefficients are

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{w}{2\pi}\left[\int_{-\pi/w}^{0} 0\,dx + \int_{0}^{\pi/w} E \sin wx\,dx\right] = \frac{w}{2\pi}\left[-\frac{E\cos wx}{w}\right]_0^{\pi/w} = \frac{w}{2\pi}\left(\frac{2E}{w}\right) = \frac{E}{\pi}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0\cos nwx\,dx + \int_{0}^{\pi/w} E \sin wx \cos nwx\,dx\right]$$

$$= \frac{wE}{2\pi}\int_{0}^{\pi/w} [\sin(1+n)wx + \sin(1-n)wx]dx = \frac{wE}{2\pi}\left[\frac{-\cos(1+n)wx}{(1+n)w} - \frac{\cos(1-n)wx}{(1-n)w}\right]_0^{\pi/w}, n \ne 1$$

$$= \frac{E}{2\pi}\left[\left(\frac{-\cos(1+n)\pi + 1}{1+n}\right) + \left(\frac{-\cos(1-n)\pi + 1}{1-n}\right)\right] = \frac{E}{2\pi}\left[\frac{-(-1)^{1+n} + 1}{1+n} + \frac{-(-1)^{1-n} + 1}{1-n}\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \quad (\text{except } n = 1) \\ \dfrac{-2E}{(n-1)(n+1)\pi}, & \text{if } n \text{ is even} \end{cases}$$

For $n = 1$, $a_1 = \dfrac{w}{\pi}\displaystyle\int_0^{\pi/w} E \sin wx \cos wx\, dx = \dfrac{wE}{2\pi}\int_0^{\pi/w} \sin 2wx\, dx = \dfrac{-E}{2\pi}\left[\dfrac{\cos 2wx}{2}\right]_0^{\pi/w} = 0.$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx = \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0 .\sin nwx\, dx + \int_0^{\pi/w} E \sin wx \sin nwx\, dx\right]$$

$$= \frac{wE}{2\pi}\int_0^{\pi/w} [\cos(1-n)wx - \cos(1+n)wx]dx$$

$$= \frac{wE}{2\pi}\left[\frac{\sin(1-n)wx}{(1-n)w} - \frac{\sin(1+n)wx}{(1+n)w}\right]_0^{\pi/w}, \quad n \neq 1$$

$$= \frac{E}{2\pi}(0) = 0.$$

For $\quad n = 1$, $b_1 = \dfrac{wE}{\pi}\displaystyle\int_0^{\pi/w} \sin wx \sin wx\, dx = \dfrac{wE}{\pi}\int_0^{\pi/w} \sin^2 wx\, dx$

$$= \frac{wE}{2\pi}\int_0^{\pi/w} (1 - \cos 2wx)dx = \frac{wE}{2\pi}\left[x - \frac{\sin 2wx}{2w}\right]_0^{\pi/w} = \frac{wE}{2\pi}\left(\frac{\pi}{w}\right) = \frac{E}{2}.$$

Hence Fourier series of $f(x)$ on $\left(\dfrac{-\pi}{w}\ \dfrac{\pi}{w}\right)$ is

$$f(x) = \frac{E}{\pi} + \frac{E}{2}\sin wx - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2wx + \frac{1}{3.5}\cos 4wx + \ldots\right].$$

Example 17.7: Obtain the Fourier series for the periodic function

$$f(x) = e^{-x}, \quad -l < x < l, \text{ where } f(x + 2l) = f(x).$$

Solution: The function $f(x)$ is periodic with period $2l$. The Fourier coefficients are

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{1}{2l}\int_{-l}^{l} e^{-x}dx = \frac{1}{2l}[-e^{-x}]_{-l}^{l} = \frac{1}{2l}(e^{l} - e^{-l}) = \frac{\sinh l}{l},$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi x}{l}+\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[-e^{-l}\cos n\pi + e^{l}\cos n\pi\right]$$

$$= \frac{2l\cos n\pi}{l^2+n^2\pi^2}\left(\frac{e^l-e^{-l}}{2}\right) = \frac{2(-1)^n l}{l^2+n^2\pi^2}\sinh l$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l}-\frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= -\frac{l}{l^2+n^2\pi^2}\left[\frac{n\pi}{l}(e^{-l}-e^{l})\cos n\pi\right]$$

$$= \frac{2n\pi\cos n\pi}{l^2+n^2\pi^2}\left(\frac{e^l-e^{-l}}{2}\right) = \frac{2(-1)^n n\pi}{l^2+n^2\pi^2}\sinh l$$

Hence the Fourier series of $f(x)$ on $(-l, l)$ is

$$e^{-x} = \frac{\sinh l}{l} + 2\sinh l\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{l^2+n^2\pi^2}\left(l\cos\frac{n\pi x}{l}+n\pi\sin\frac{n\pi x}{l}\right)\right)$$

$$= \sinh l\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^3\pi^2}\cos\frac{3\pi x}{l} - \ldots\right)\right.$$

$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \ldots\right)\right]$$

EXERCISE 17.1

1. Obtain the Fourier series to represent

$$f(x) = \frac{1}{4}(\pi - x)^2, \quad 0 < x < 2\pi, \; f(x + 2\pi) = f(x).$$

2. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -\pi < x \le 0 \\ -1, & 0 < x < \pi \end{cases}, \quad f(x + 2\pi) = f(x).$$

3. Find the Fourier series expansion of the function

$$f(x) = x \sin x, \quad 0 < x < 2\pi, \; f(x + 2\pi) = f(x).$$

4. Find the Fourier series to represent $f(x) = x - x^2$, $-\pi < x < \pi$. Hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$

5. Obtain the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$

6. Let f be the periodic function shown in the Fig. 17.9, each segment of which is a semicircle of radius π. Show that its Fourier series expansion is

$$f(x) = \frac{\pi^2}{4} + \frac{\pi^2}{2} \sum_{n=1}^{\infty} [J_0(n\pi) + J_2(n\pi)] \cos nx,$$

where J_0 and J_2 have their usual meanings.

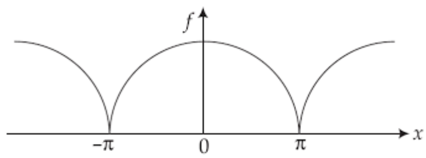


Fig. 17.9

7. Find the Fourier series expansion of the function

$$f(x) = e^{-4x}, \quad -2 \le x \le 2, \quad f(x + 4) = f(x).$$

8. Find the Fourier series of the *periodic square wave* given by

$$f(x) = \begin{cases} 0, & -2 < x \le -1 \\ k, & -1 < x < 1 \\ 0, & 1 \le x < 2 \end{cases} \quad f(x + 4) = f(x).$$

9. Find the Fourier series of the periodic function

$$f(x) = \pi \sin \pi x, \quad 0 < x < 1, \quad f(x + 1) = f(x).$$

10. Prove that in the range $-\pi < x < \pi$,

$$\cosh ax = \frac{2a^2}{\pi} \sinh a\pi \left(\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} \cos nx \right).$$

17.3 FOURIER SERIES EXPANSIONS OF EVEN AND ODD FUNCTIONS

We can save some work in computing the Fourier coefficients, if a function is even or odd. Before proceeding further, first we discuss the concept of even and odd functions and a few of their properties from calculus.

17.3.1 Even and Odd Functions

A function f(x) is an even function on [–l, l], if $f(-x) = f(x)$, for $-l \le x \le l$.

For example, $y = x^2, x^4, \cos x, e^{-|x|}$ are even functions of x on any interval $[-l, l]$. The graph of such a function is symmetric with respect to the y-axis as shown in Fig. 17.10 for $y = x^2$.

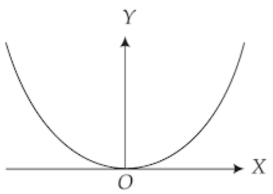


Fig. 17.10

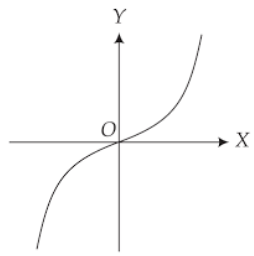


Fig. 17.11

A function g(x) is an odd function on [–l, l], if $g(-x) = -g(x)$, *for* $-l \le x \le l$.

For example, $y = x, x^3, \sin x$ are odd functions of x on any interval $[-l, l]$. The graph of such a function is symmetric with respect to the orgin as shown in Fig. 17.11 for $y = x^3$.

A function may neither be an even function nor an odd function. For example, functions $y = x + x^2, e^x$ are not even nor odd on any interval $[-l, l]$.

Further, the product of two even functions or two odd functions is an even function and the product of an even function with an odd function is an odd function. Also we have following results from calculus

1. $$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx, \quad \text{...(17.19)}$$

if f is an even function of x on $[-l, l]$, and

2. $$\int_{-l}^{l} f(x)dx = 0, \qquad ...(17.20)$$

if f is an odd function on $[-l, l]$.

17.3.2 Fourier Series Expansions of Even and Odd Functions

We have already observed in Example (17.1) with $f(x) = x$, which is an odd function of x on $[-\pi, \pi]$, that the cosine coefficients were all zeros, since $x \cos nx$ is an odd function of x and thus the Fourier expansion of $f(x) = x$ consists only sine terms. We have the following results for the Fourier series expansion of even and odd functions:

Theorem 17.2: *If $f(x)$ is an even function on $[-l, l]$ of period $2l$, then*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} dx \qquad ...(17.21)$$

with coefficients

$$a_0 = \frac{1}{l}\int_0^l f(x)dx \text{ and } a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l} dx. \qquad ...(17.22)$$

Theorem 17.3: *If $f(x)$ is an odd function on $[-l, l]$ of period $2l$, then*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} dx, \qquad ...(17.23)$$

with coefficients

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx. \qquad ...(17.24)$$

These results follow easily from the applications of (17.19) and (17.20) to the Euler's formulae for the Fourier coefficients given by (17.18).

The series in (17.21) is called the *Fourier cosine series* and the series in (17.23) is called the *Fourier sine series*.

When the period is 2π, then (17.21) reduces to

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad ...(17.25)$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx; \qquad ...(17.26)$$

and (17.23) reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad \text{...(17.27)}$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx. \qquad \text{...(17.28)}$$

Example 17.8: Find the Fourier series of $f(x) = x^2$ on $(-1, 1)$, when $f(x + 2) = f(x)$.

Solution: Since the function $f(x) = x^2$ is an even function of x on $[-1, 1]$, thus its Fourier series expansion consists only of constant term and cosine terms; also $f(x)$ is periodic with period 2. The Fourier coefficients are

$$a_0 = \frac{1}{l}\int_0^l f(x)dx = \int_0^1 x^2 dx = \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3},$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = 2\int_0^1 x^2 \cos n\pi x\, dx = 2\left[\left[x^2\frac{\sin n\pi x}{n\pi}\right]_0^1 - 2\int_0^1 x\frac{\sin n\pi x}{n\pi}dx\right]$$

$$= -\frac{4}{n\pi}\left[-x\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 = \frac{4}{n^2\pi^2}\cos n\pi = \frac{4(-1)^n}{n^2\pi^2}.$$

Hence, the Fourier series for $f(x)$ on $(-1, 1)$ is

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

Example 17.9: Find the Fourier series expansion of the function $f(x) = \sin ax$, $-\pi < x < \pi$, where a is not an integer.

Solution: Since $\sin ax$ is an odd function of x on $[-\pi, \pi]$, thus its Fourier series expansion consists of only sine terms. The Fourier coefficients are

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^{\pi}\sin ax \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi}[\cos(n-a)x - \cos(n+a)x]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{(n+a)}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{(n-a)} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = (-1)^{n+1}\frac{2n\sin a\pi}{\pi(n^2-a^2)}.$$

Hence the Fourier series for $f(x)$ on $(-\pi, \pi)$ is

$$\sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2-a^2}\sin nx.$$

Example 17.10: Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1+2x/\pi, & -\pi < x < 0 \\ 1-2x/\pi, & 0 \le x < \pi \end{cases} \quad f(x+2\pi) = f(x).$$

Also deduce that $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{8}.$$

Solution: The graph of the function $f(x)$ is shown in Fig. 17.12. The graph is symmetrical about y-axis. Hence, $f(x)$ is an even function of x over $(-\pi, \pi)$ with period 2π. Thus its Fourier expansion consists of only constant term and cosine terms. The Fourier coefficients are

$$a_0 = \frac{1}{l}\int_0^l f(x)dx = \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{1}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi} = 0,$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

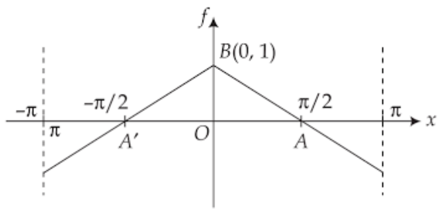


Fig. 17.12

$$= \frac{2}{\pi}\left[\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{2}{\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{4}{\pi^2}\left[\frac{-\cos nx}{n^2}\right]_0^{\pi} = \frac{4}{n^2\pi^2}(1-\cos n\pi) = \frac{4}{n^2\pi^2}(1-(-1)^n).$$

Thus, $$a_n = \begin{cases}\dfrac{8}{n^2\pi^2}, & \text{for odd } n \\ 0, & \text{for even } n\end{cases}.$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]. \qquad \ldots(17.29)$$

At $x = 0$, $f(0) = 1$. Setting $x = 0$ in (17.29), we obtain

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right), \quad \text{or} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

17.4 FOURIER HALF-RANGE COSINE AND SINE SERIES

We have seen that in case $f(x)$ is defined on $-l \le x \le l$ we can write its Fourier series, and the coefficients of the series are determined by the function and the interval. Let us suppose that a function $f(x)$ of period $2l$ is specified only on a half-range interval $0 \le x \le l$. In such a case, we have a choice to extend the definition of the function to the interval $-l \le x \le l$ in a suitable manner, even or odd, to find its Fourier cosine or sine expansion respectively and then restricting the Fourier series representation of the extended function to the original half-range interval $0 \le x \le l$.

17.4.1 The Fourier Cosine Series on $0 \le x \le l$

Let a function $f(x)$ specified on the interval $0 \le x \le l$ is extended to the interval $-l \le x \le l$ as an even function $g(x)$ of x, given by

$$g(x) = \begin{cases} f(-x), & -l \le x \le 0 \\ f(x), & 0 \le x \le l \end{cases}$$

which coincides with $f(x)$ on the interval $0 \le x \le l$, refer Fig. 17.13a and 17.13b, then

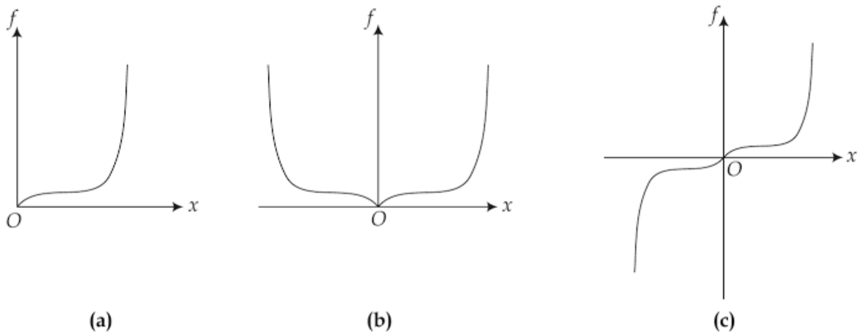


Fig. 17.13

the Fourier series representation of $f(x)$ on the interval $0 \le x \le l$ is the cosine series given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad \ldots(17.30)$$

where $$a_0 = \frac{1}{l}\int_0^l f(x)dx \text{ and } a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx.$$

17.4.2 The Fourier Sine Series on $0 \le x \le l$

If $f(x)$ specified on the interval $0 \le x \le l$, is extended to the interval $-l \le x \le l$ as an odd function $g(x)$ of x, given by

$$g(x) = \begin{cases} -f(-x), & -l \le x \le 0 \\ f(x), & 0 \le x \le l \end{cases}$$

which coincides with $f(x)$ on the interval $0 \le x \le l$, refer Figs. 17.13a and 17.13c, then the Fourier series representation of $f(x)$ on the interval $0 \le x \le l$ is the sine series given by

$$f(x) = \sum_{n=1}^{\infty}{}' b_n \sin\frac{n\pi x}{l}, \qquad \text{...(17.31)}$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx.$$

The expansions (17.30) and (17.31) are respectively referred to as *half-range cosine series* and *half-range sine series* expansions of $f(x)$.

Example 17.11: Find the Fourier sine and cosine series expansions of $f(x) = x$ for $0 \le x \le \pi$.

Solution: The sine series representation of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty}{}' b_n \sin nx; \quad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx.$$

Substituting for $f(x)$, we have

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi = -\frac{2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n}.$$

Hence, the required sine series expansion of $f(x) = x$ is

$$x = 2\sum_{n=1}^{\infty}{}' (-1)^{n+1}\frac{\sin nx}{n}, \; 0 \le x \le \pi.$$

Next, the cosine series representation of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}{}' a_n \cos nx; \quad a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx \text{ and } a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx.$$

Substituting for $f(x)$, we have

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi/2, \text{ and}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2}[(-1)^n - 1].$$

Thus,
$$a_n = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{when } n \text{ is odd.} \\ 0, & \text{when } n \text{ is even.} \end{cases}$$

Hence the cosine series expansion of $f(x) = x$ is

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \quad 0 \le x \le \pi.$$

Example 17.12: Write the sine series expansion of

$$f(x) = \begin{cases} 1, & 0 < x \le \pi/2 \\ 2, & \pi/2 < x < \pi \end{cases}$$

on $[0, \pi]$ and also discuss its convergence.

Solution: The sine series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \text{ with } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Substituting for $f(x)$, we have

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\sin nx\,dx + \int_{\pi/2}^{\pi} 2\sin nx\,dx\right] = \frac{2}{\pi}\left[\left[-\frac{\cos nx}{n}\right]_0^{\pi/2} + 2\left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{n\pi}\left[\left(1 - \cos\frac{n\pi}{2}\right) - 2\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right] = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right].$$

Hence, the sine series expansion of $f(x)$ is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right)\sin nx.$$

The series converges to 0, for $x = 0$; to 1, for $0 < x < \pi/2$; to $\frac{1}{2}(1 + 2) = 3/2$, for $x = \pi/2$; and to 2, for $\pi/2 < x < \pi$, and again to 0 for $x = \pi$.

EXERCISE 17.2

1. Expand the function $f(x) = x^4$ on $[-1, 1]$ as a Fourier series.
2. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $[-\pi, \pi]$. Also deduce that
$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots = \frac{1}{4}(\pi - 2).$$
3. Expand the function $f(x) = |x|$, $-\pi < x < \pi$ as Fourier series and hence deduce that
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$
4. Expand $f(x) = |\cos x|$, $-\pi < x < \pi$ as a Fourier series.
5. Expand $f(x) = \begin{cases} -1/2, & -\pi < x < 0 \\ 1/2, & 0 < x < \pi \end{cases}$ $\quad f(x + 2\pi) = f(x)$

   as a Fourier series.
6. Expand $f(x) = \begin{cases} -x + 1, & -\pi \le x \le 0 \\ x + 1, & 0 \le x \le \pi \end{cases}$ $\quad f(x + 2\pi) = f(x)$

   as a Fourier series. Also deduce the value of $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$
7. Obtain the Fourier series expansion of $f(x) = 4 - x^2$, $-2 \le x \le 2$. Also show that
$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$
8. Find the Fourier cosine and sine series of $f(x) = 1$, $0 \le x \le 2$.
9. Find the Fourier cosine series of the function
$$f(x) = \begin{cases} x^2, & 0 \le x < 2 \\ 4, & 2 \le x \le 4. \end{cases}$$
10. Find the Fourier sine series of the function
$$f(x) = \begin{cases} x, & 0 \le x < \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases}$$
11. Expand $\sin\left(\dfrac{\pi x}{l}\right)$ in half-range cosine series in the interval $[0, l]$.
12. Find the Fourier cosine series for the function
$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & \pi/2 < x \le \pi. \end{cases}$$

17.5 INTEGRATION AND DIFFERENTIATION OF FOURIER SERIES. THE PARSEVAL'S FORMULA

First we discuss the termwise integration and differentiation of the Fourier series of a function $f(x)$ and then using the concept of termwise integration, we derive the Parseval's formula.

17.5.1 Termwise Integration and Differentiation of Fourier Series

Let $f(x)$ be piecewise continuous on $[-l, l]$ with Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right).$$

Then for any x, $-l \le x \le l$,

$$\int_{-l}^{x} f(t)dt = a_0(x + l) + \frac{l}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin \frac{n\pi x}{l} - b_n\left(\cos \frac{n\pi x}{l} - (-1)^n\right)\right]. \qquad ...(17.32)$$

Note that the expression on the right side of (17.32) is exactly what we get by integrating the Fourier series term by term from $-l$ to x. This holds even if the Fourier series does not converge to $f(x)$ at a particular value of x.

Example 17.13: Use the Fourier series representation of $f(x) = x$, $-\pi < x < \pi$ to find the Fourier series representation for x^2 over $-\pi < x < \pi$.

Solution: The Fourier series representation of the function $f(x) = x$ over $[-\pi, \pi]$, refer Example 17.1 is

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx. \qquad ...(17.33)$$

Integrating (17.33) term by term over the interval $(-\pi, x)$ for any x in $-\pi < x < \pi$, we obtain

$$\int_{-\pi}^{x} x\,dx = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \int_{-\pi}^{x} \sin nx\,dx,$$

or, $$\frac{1}{2}(x^2 - \pi^2) = \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n[\cos nx - \cos n\pi],$$

or, $$x^2 - \pi^2 = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\cos nx - (-1)^n],$$

$$= 4 \sum_{n=1}^{\infty}{}' \frac{(-1)^n}{n^2} \cos nx - 4 \sum_{n=1}^{\infty}{}' \frac{1}{n^2}. \quad ...(17.34)$$

Using the result $\sum_{n=1}^{\infty}{}' \frac{1}{n^2} = \frac{\pi^2}{6}$, (refer, Problem 5 Exercise 17.1), (17.34) becomes

$$x^2 - \pi^2 = 4 \sum_{n=1}^{\infty}{}' \frac{(-1)^n}{n^2} \cos nx - \frac{2\pi^2}{3},$$

or,
$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty}{}' \frac{(-1)^n}{n^2} \cos nx,$$

the Fourier series representation of x^2 over $-\pi \le x \le \pi$.

We have seen that termwise integration of Fourier series of a function $f(x)$ leads to some meaningful results. But same is not always true in case of termwise differentiation,

Consider again the Fourier series expansion of x over the interval $-\pi \le x \le \pi$. It is

$$x = \sum_{n=1}^{\infty}{}' \frac{2}{n} (-1)^{n+1} \sin nx. \quad ...(17.35)$$

The series converges to x for $-\pi < x < \pi$.

Differentiating w.r.t. x for $-\pi < x < \pi$, we obtain

$$1 = \sum_{n=1}^{\infty}{}' 2(-1)^{n+1} \cos nx, \quad ...(17.36)$$

which is absurd, since the right side of (17.36) does not even converge over $-\pi < x < \pi$.

Thus, in this case termwise derivative of Fourier series is not related to the derivative of $f(x)$. However, for the validity of termwise differentiation of Fourier series, the series should be uniformly convergent over the given interval which is not true in case of the right side of (17.35).

Example 17.14: Use the Fourier series representation of $f(x) = x^2$, $-\pi < x < \pi$ to find the Fourier series representation for x over $-\pi < x < \pi$.

Solution: The Fourier series representation of the function $f(x) = x^2$ over $[-\pi, \pi]$, refer Example (17.13), is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty}{}' \frac{(-1)^n}{n^2} \cos nx. \quad ...(17.37)$$

The series on the right side of (17.37) is uniformly convergent over $-\pi < x < \pi$, thus termwise differentiation of (17.37) is admissible. Then for $-\pi < x < \pi$

$$f'(x) = 2x = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

or,
$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx,$$

which is the Fourier series representation of x over $-\pi < x < \pi$.

17.5.2 The Parseval's Formula

We state the following result:

Theorem 17.4 (Parseval's formula): *If the Fourier series for $f(x)$ converges uniformly on $(-l, l)$, then*

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2l}\int_{-l}^{l} [f(x)]^2 dx, \qquad ...(17.38)$$

where a_0, a_n, b_n, $n = 1, 2, ...$ are the Fourier coefficients of f on $(-l, l)$.

Proof. The Fourier series for $f(x)$ on $(-l, l)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

Since the series is uniformly convergent over $(-l, l)$, multiplying both sides of it by $f(x)$ and integrating termwise from $-l$ to l, we get

$$\int_{-l}^{l} [f(x)]^2 dx = a_0 \int_{-l}^{l} f(x)dx + \sum_{n=1}^{\infty} \left(a_n \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx + b_n \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right)$$

$$= l\left[2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right], \quad \text{using (17.18)}$$

or,
$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2l}\int_{-l}^{l} [f(x)]^2 dx, \text{ which is (17.38).}$$

In case the interval is $(0, 2l)$, then the Parseval formula corresponding to (17.38), is

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2l}\int_{0}^{2l} [f(x)]^2 dx. \qquad ...(17.39)$$

On the similar lines we can prove the following two results corresponding to half-range expansion of $f(x)$ over the interval $[0, l]$.

Theorem 17.5: *If half-range cosine series of $f(x)$ converges uniformly to $f(x)$ over $(0, l)$, then*

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx, \qquad \text{...(17.40)}$$

where $\quad a_0 = \dfrac{1}{l}\displaystyle\int_0^l f(x)dx$ and $a_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\cos\frac{n\pi x}{l}dx.$

Theorem 17.6: *If the half-range sine series of $f(x)$ converges uniformly to $f(x)$ over $(0, l)$, then*

$$\frac{1}{2}\sum_{n=1}^{\infty} b_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx, \qquad \text{...(17.41)}$$

where $\quad b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}dx.$

The expression

$$[f(x)]_{rms} = \sqrt{\frac{1}{2l}\int_{-l}^{l} [f(x)]^2 dx} \qquad \text{...(17.42)}$$

is called the *root mean square (r.m.s.) value of the function $f(x)$ over the interval $(-l, l)$*. The *r.m.s.* value of a periodic function finds applications in engineering physics.

Example 17.15: Find the Fourier series expansion of the function $f(x) = |x|$ defined over the interval $(-2, 2)$. Using Parseval equality, prove that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}.$$

Solution: Since $f(x)$ is an even function of $f(x)$ over the interval $(-2, 2)$, thus the Fourier series expansion of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}, \qquad \text{...(17.43)}$$

We have $\quad a_0 = \dfrac{1}{2}\displaystyle\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx = \int_0^2 x\cos\frac{n\pi x}{2}dx = \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \frac{4}{n^2\pi^2}[(-1)^n - 1].$$

Hence, (17.43) becomes

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\frac{n\pi x}{2}.$$

Applying the Parseval equality

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx,$$

we obtain

$$1^2 + \frac{1}{2}\cdot\frac{16}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]^2 = \frac{1}{2}\int_0^2 x^2 dx$$

or, $$1 + \frac{32}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right) = 4/3,$$ which gives

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}.$$

Example 17.16: Using the Fourier coefficients of $f(x) = \cos(x/2)$ on $(-\pi, \pi)$ prove that

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2} = \frac{\pi^2 - 8}{16}.$$

Solution: The function $f(x) = \cos(x/2)$ is an even function of x on the interval $(-\pi, \pi)$. The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \cos\frac{x}{2}\,dx = \frac{2}{\pi}\left[\sin\frac{x}{2}\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\left[\cos\left(n + \frac{1}{2}\right)x + \cos\left(n - \frac{1}{2}\right)x\right]dx$$

$$= \frac{2}{\pi}\left[\frac{\sin\left(n + \frac{1}{2}\right)x}{2n+1} + \frac{\sin\left(n - \frac{1}{2}\right)x}{2n-1}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\sin\left(n + \frac{1}{2}\right)\pi}{2n+1} + \frac{\sin\left(n - \frac{1}{2}\right)\pi}{2n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1}\right] = \frac{-4(-1)^n}{\pi(4n^2 - 1)}.$$

By Parseval formula

$$a_0^2 + \frac{1}{2}\sum a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx.$$

Substituting for a_0, a_n, l and $f(x)$, we get

$$\frac{4}{\pi^2} + \frac{16}{2\pi^2}\sum \frac{1}{(4n^2-1)^2} = \frac{1}{\pi}\int_0^{\pi} \cos^2 (x/2)\, dx = \frac{1}{2\pi}\int_0^{\pi}(1+\cos x)\, dx = \frac{1}{2}$$

$$\text{or,} \quad \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}.$$

EXERCISE 17.3

1. If $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$ has the Fourier series representation, $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin (2n-1)x}{2n-1}$,

then find the Fourier series representation of $g(x) = \begin{cases} -x-\pi, & -\pi < x < 0 \\ x-\pi, & 0 < x < \pi. \end{cases}$

2. Find the Fourier series of $f(x) = \pi^2 - x^2$ for $-\pi < x < \pi$ and use it to find the Fourier series of x and $x(\pi^2 - x^2)$.

3. Let $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x, & 0 < x \le \pi \end{cases}$. Write the Fourier series of $f(x)$ on $[-\pi, \pi]$ and show that this series converges to $f(x)$ on $(-\pi, \pi)$, and can be integrated term by term, and thus obtain a trigonometric series expansion for $g(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x^2/2, & 0 < x \le \pi. \end{cases}$

4. If $f(x) = x \sin x$, $-\pi < x < \pi$ with the Fourier series representation

$$f(x) = \pi - \frac{1}{2}\pi \cos x + 2\pi\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos nx,$$

show that the series can be differentiated term by term and hence find the Fourier series expansion for $g(x) = x \cos x + \sin x$, for $-\pi < x < \pi$.

5. Given $f(x) = \begin{cases} \sin 2x, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x \le \pi/2 \\ \sin 2x, & \pi/2 < x \le \pi \end{cases}$

Find the Fourier series expansion for $f'(x)$ by differentiating the Fourier expansion for $f(x)$.

6. Using the Fourier coefficients for the function $f(x) = \begin{cases} -1, & -\pi < x \le 0 \\ 1, & 0 < x < \pi \end{cases}$, show that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

7. Using the Fourier coefficients for the function $y = x^2$ on $[-\pi, \pi]$, show that $\dfrac{\pi^4}{90} = \sum_{n=1}^{\infty} \dfrac{1}{n^4}$.

8. From the coefficients of the half-range cosine series of the function

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}, \quad \text{find } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

17.6 COMPLEX FORM OF THE FOURIER SERIES

To simplify the calculations it is sometimes convenient to work in terms of complex numbers even when the parameters under reference are reals. In this context we study the complex form of the Fourier series of a real function $f(x)$. This form is of special interest in the study of electrical circuits.

17.6.1 Complex Fourier Series

Let $f(x)$ be a real periodic function of period $2l$ over the interval $(-l, l)$. Then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right).$$

Since, $\cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$, therefore, this series can be expressed as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ \frac{a_n}{2}\left(e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}} \right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}} \right) \right\}$$

and after regrouping the terms, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{\frac{-in\pi x}{l}}. \qquad \text{...(17.44)}$$

We define $c_0 = a_0$, $c_n = \dfrac{a_n - ib_n}{2}$ and $c_{-n} = \dfrac{a_n + ib_n}{2}$, for $n = 1, 2, \ldots$.

Clearly c_n and c_{-n} are complex conjugates. Using these, (17.44) becomes

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{\frac{in\pi x}{l}}, \quad \text{for} \quad -l < x < l, \qquad \text{...(17.45)}$$

where
$$c_n = \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - \frac{i}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx$$

and,
$$c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{in\pi x}{l}}dx.$$

Combining these two, we have

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx, \quad n = 0, \pm 1, \pm 2. \qquad \text{...(17.46)}$$

Thus, (17.45) is the complex form of the Fourier series representation of the periodic function $f(x)$ defined over the interval $-l < x < l$, with Fourier coefficients given by (17.46).

Since, the complex form of the Fourier series representation of a function is derived from its real variable definition, the convergence properties of the complex Fourier series are the same as those for the real variable case. Thus, at points of continuity of $f(x)$ the series converges to $f(x)$, while at points of discontinuity it converges to the mid-point.

Example 17.17: Find the complex Fourier series representation of the function

$$f(x) = \begin{cases} 0, & 0 < x \le 1 \\ 1, & 1 < x < 4 \end{cases}$$

when $f(x) = f(x+4)$.

Solution: The function $f(x)$ is periodic with period 4 defined on the interval (0, 4), with $2l = 4$. Thus the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{4}\int_0^4 f(x)e^{\frac{-in\pi x}{2}}dx = \frac{1}{4}\int_1^4 e^{\frac{-in\pi x}{2}}dx.$$

For $n = 0$, we get

$$c_0 = \frac{1}{4}\int_1^4 dx = 3/4.$$

For all c_n, except $n = 0$

$$c_n = \frac{1}{4}\left[\frac{-2}{in\pi}e^{\frac{-in\pi x}{2}}\right]_1^4 = \frac{i}{2\pi n}[1 - e^{-in\pi/2}].$$

Hence, the complex Fourier series representation of $f(x)$ is

$$f(x) = \frac{3}{4} + \lim_{k\to\infty}\sum_{n=-k}^{k}\frac{i}{2\pi n}(1 - e^{-in\pi/2})e^{\frac{in\pi x}{2}}, \quad (n \neq 0).$$

Example 17.18: Find the complex Fourier series representation of the function

$$f(x) = e^{-x}, \quad -\pi < x < \pi; \quad f(x) = f(x + 2\pi)$$

Solution: The function $f(x)$ is periodic with period 2π, defined on the interval $(-\pi, \pi)$. Here $l = \pi$, thus the complex Fourier coefficients are

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(1+in)x}\,dx \\
&= \frac{-1}{2\pi(1+in)}[e^{-(1+in)x}]_{-\pi}^{\pi} = \frac{-1}{2\pi(1+in)}[e^{-(1+in)\pi} - e^{(1+in)\pi}] \\
&= \frac{-1}{2\pi(1+in)}[e^{-\pi}(\cos n\pi - i\sin n\pi) - e^{\pi}(\cos n\pi + i\sin n\pi)] \\
&= \frac{(1-in)}{2\pi(1+n^2)}[(e^{\pi} - e^{-\pi})\cos n\pi] = (-1)^n\frac{(1-in)\sinh\pi}{\pi(1+n^2)}.
\end{aligned}$$

Hence, the complex Fourier series is

$$f(x) = \frac{\sinh\pi}{\pi}\lim_{k\to\infty}\sum_{n=-k}^{k}(-1)^n\left(\frac{1-in}{1+n^2}\right)e^{inx}.$$

17.6.2 Frequency Spectra of a Function $f(x)$

In applications of Fourier series to periodic physical phenomena with fundamental period T, it is sometimes more convenient to work in terms of the angular frequency w, defined as

$$w = 2\pi/T = 2\pi/2l = \pi/l,$$

called the *fundamental angular frequency*. In terms of w, the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right),$$

can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nwx + b_n \sin nwx), \qquad ...(17.47)$$

where $$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos nwx\,dx \text{ and } b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin nwx\,dx; \quad l = \pi/w.$$

In terms of the fundamental angular frequency, the complex Fourier series form (17.45) can be expressed as

$$f(x) = \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{inwx}, \qquad ...(17.48)$$

where $$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-inwx}dx, \quad l = \pi/w, \; n = 0, \; \pm 1, \; \pm 2, \; ... \qquad ...(17.49)$$

The plot of the points $(nw, |c_n|)$, where w is the fundamental angular frequency and c_n are the Fourier coefficients as defined in (17.49) is called the *frequency spectrum* or *amplitude spectrum* of the function $f(x)$ and the number nw is called the *nth harmonic frequency* of the function $f(x)$.

Example 17.19: Find the frequency spectrum of the periodic pulse defined by

$f(x) = 3x/4, \quad 0 \le x \le 8$ and $\quad f(x + 8) = f(x)$.

Solution: The function $f(x)$ is periodic with period $T = 2l = 8$ defined on [0, 8]. Thus the fundamental angular frequency w is

$$w = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}.$$

The complex Fourier coefficients are

$$c_n = \frac{1}{8}\int_0^8 \frac{3}{4}xe^{-\frac{in\pi}{4}x}\,dx = \frac{3}{32}\left[x\left(\frac{4}{-in\pi}\right)e^{-\frac{in\pi}{4}x} - \left(\frac{4}{-in\pi}\right)^2 e^{-\frac{in\pi x}{4}}\right]_0^8$$

$$= \frac{3i}{n\pi}, \quad n \ne 0, \text{ after simplification.}$$

For $n = 0$, $\quad c_0 = \dfrac{3}{32}\displaystyle\int_0^8 x\,dx = \frac{3}{32}\left(\frac{x^2}{2}\right)_0^8 = 3.$

The frequency spectrum of $f(x)$ is a plot of points $(nw, |c_n|)$, where

$$nw = \frac{n\pi}{4}, \quad |c_0| = 3 \text{ and } |c_n| = \frac{3}{|n|\pi}, \text{ for } n = \pm 1, \pm 2, \pm 3, \ldots .$$

The plot is shown in Fig. 17.14.

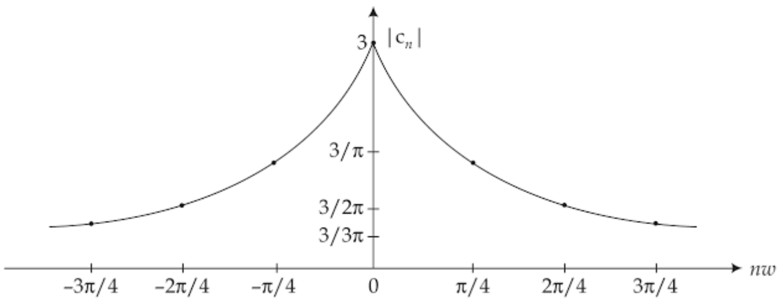


Fig. 17.14

Example 17.20: Find the frequency spectrum of the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

when $f(x + 2\pi) = f(x)$, for all x.

Solution: The function $f(x)$ is periodic with period $T = 2\pi$ defined over the interval $(-\pi, \pi)$. The fundamental angular frequency w is

$$w = 2\pi/T = 2\pi/2\pi = 1.$$

The complex Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx.$$

For $n = 0$, $c_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi/2}^{\pi/2} dx = \dfrac{1}{2}$; and for all other $n \neq 0$

$$c_n = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx = \frac{1}{n\pi}\left[\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right] = \frac{1}{n\pi}\sin\frac{n\pi}{2}, \quad n = \pm 1, \pm 2, \ldots.$$

The frequency spectrum of $f(x)$ is a plot of points $(nw, |c_n|)$. Here

$$nw = n,\ |c_o| = \frac{1}{2} \text{ and } |c_n| = \frac{1}{n\pi}\left|\sin\frac{n\pi}{2}\right|, \text{ for } n = \pm 1, \pm 2, \dots .$$

The plot is as shown in Fig. 17.15.

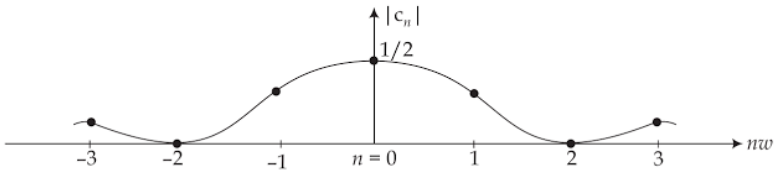


Fig. 17.15

EXERCISE 17.4

Find the complex Fourier series representation of $f(x)$ on the given interval

1. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$
2. $f(x) = e^x, \quad 0 < x < 1, \quad f(x+1) = f(x).$
3. $f(x) = |E \sin \lambda x|, \quad 0 < x < \pi/\lambda, \quad f(x + \pi/\lambda) = f(x)$
4. $f(x) = e^{-|x|}, \quad -2 < x < 2, \quad f(x+4) = f(x)$

Find the frequency spectrum of $f(x)$ for the following problems

5. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \sin x, & 0 \le x < \pi/2 \end{cases}, \quad f(x+\pi) = f(x)$
6. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$
7. $f(x) = |E \sin \lambda x|, \quad 0 < x < \pi/\lambda, \quad f(x + \pi/\lambda) = f(x)$
8. Plot some points of the frequency spectrum of the function defined by

$$f(x) = 4 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{26}{n(12-5i)} e^{2nix}.$$

17.7 NUMERICAL HARMONIC ANALYSIS

So far we have derived the Fourier series expansion of a function $f(x)$ when it was known analytically. However, in many practical problems the analytic nature of the periodic function $f(x)$

is not known but one may be in a position to observe only a set of values of x and y; y being dependent on x, say $y = f(x)$. In such a case, to evaluate the Fourier coefficients, the Euler's formulae studied previously need some modifications given as below.

Let (x_i, y_i), $i = 1, 2, \ldots k$ be the given set of values, where the x_i's are equally spaced. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} y dx$$

$$= [\text{Mean value of } y \text{ over the one period } T = 2\pi]$$

$$= \frac{1}{k}\sum_{i=1}^{k} y_i, \qquad \text{...(17.50)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2.\frac{1}{2\pi}\int_{-\pi}^{\pi} y\cos nx\,dx$$

$$= 2[\text{Mean value of } y\cos nx \text{ over the one period } T = 2\pi]$$

$$= \frac{2}{k}\sum_{i=1}^{k} y_i \cos nx_i, \qquad \text{...(17.51)}$$

and,
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2.\frac{1}{2\pi}\int_{-\pi}^{\pi} y\sin nx\,dx$$

$$= 2[\text{Mean value of } y \sin \text{nx over the one period } T = 2\pi]$$

$$= \frac{2}{k}\sum_{i=1}^{k} y_i \sin nx_i. \qquad \text{...(17.52)}$$

Then the Fourier series for $y = f(x)$ is

$$y = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad \text{...(17.53)}$$

where the Fourier coefficients are given by (17.50), (17.51) and (17.52).

The process of finding the Fourier series for a function given by numerical values is known as *numerical harmonic analysis*. The term $(a_1 \cos x + b_1 \sin x)$ is called *the fundamental* or *first harmonic*, the term $(a_2 \cos 2x + b_2 \sin 2x)$ is called *the second harmonic* and so on.

Example 17.21: Given that x is a function of θ over the interval $0 \le \theta \le 2\pi$. Find the Fourier series expansion of x upto the second harmonic on the basis of the following data

θ :	0,	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$
x :	298	356	373	337	254	155	80	51	60	93	147	221

Solution: The Fourier series for $x = f(\theta)$ upto the second harmonic is

$$x \approx a_0 + \sum_{n=1}^{2}{}' (a_n \cos n\theta + b_n \sin n\theta),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{12}\sum_{i=1}^{12}{}' x_i, \quad a_n = \frac{1}{6}\sum_{i=1}^{12}{}' x_i \cos n\,\theta_i, \text{ and } \quad b_n = \frac{1}{6}\sum_{i=1}^{12}{}' x_i \sin n\,\theta_i.$$

To evaluate the coefficients we form the following table:

θ	sin θ	cos θ	sin 2θ	cos 2θ	x	x sin θ	x cos θ	x sin 2θ	x cos 2θ
0	0.00	1.00	0.00	1.00	298	0.00	298.00	0.00	298.00
$\pi/6$	0.50	0.87	0.87	0.50	356	178.00	309.72	309.72	178.00
$\pi/3$	0.87	0.50	0.87	−0.50	373	324.51	186.50	324.51	−186.50
$\pi/2$	1.00	0.00	0.00	−1.00	337	337.00	0.00	0.00	−337.00
$2\pi/3$	0.87	−0.50	−0.87	−0.50	254	220.98	−127.00	−220.98	−127.00
$5\pi/6$	0.50	−0.87	−0.87	−0.50	155	77.50	−134.85	−134.85	−77.50
π	0.00	−1.00	0.00	1.00	80	0.00	−80.00	0.00	80.00
$7\pi/6$	−0.50	−0.87	0.87	0.50	51	−25.50	−44.37	44.37	25.50
$4\pi/3$	−0.87	−0.50	0.87	−0.50	60	−52.20	−30.00	52.20	−30.00
$3\pi/2$	−1.00	0.00	0.00	−1.00	93	−93.00	0.00	0.00	−93.00
$5\pi/3$	−0.87	0.50	−0.87	−0.50	147	−102.90	73.50	−102.90	−73.50
$11\pi/6$	−0.50	0.87	−0.87	0.50	221	−110.50	192.27	−192.27	110.50
				Total:	2425	753.89	643.77	54.18	−77.50

We have,

$$\Sigma x = 2425, \quad \Sigma x \sin\theta = 753.89, \quad \Sigma x \cos\theta = 643.77, \quad \Sigma x \sin 2\theta = 54.18, \quad \Sigma x \cos 2\theta = -77.50$$

Thus, $a_0 = \dfrac{2425}{12} = 202.09 \approx 202, \quad a_1 = \dfrac{643.77}{6} = 107.30 \approx 107, \quad a_2 = \dfrac{-77.50}{6} = -12.92 \approx -13.$

$$b_1 = \frac{753.89}{6} = 125.65 \approx 126 \text{ and } b_2 = \frac{54.18}{6} = 9.03 \approx 9.$$

Hence the Fourier series expansion upto two harmonics is given by

$$x \approx 202 + 107\cos\theta - 13\cos 2\theta + 126\sin\theta + 9\sin 2\theta$$

Example 17.22: The turning moment T units of the crank shaft of a steam engine is given for a series of values of the crank-angle θ in degrees;

$\theta°$	:	0	30	60	90	120	150	180
T	:	0	5224	8097	7850	5499	2626	0

Find the first four terms in a series of sines to represent T and calculate T for $\theta = 75°$.

Solution: The half-range sine series to represent T is

$$T \approx b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta,$$

where the coefficients b_i's are given by

$$b_n = \frac{2}{6}\Sigma T \sin n\theta = \frac{1}{3}\Sigma T \sin n\theta.$$

To calculate b_i's we form the following table:

θ	T	sinθ	sin 2θ	sin 3θ	sin 4θ	T sin θ	T sin 2θ	T sin 3θ	T sin 4θ
0	0	0.00	0.00	0.00	0.00	0.00	0.00	0.0	0.00
30	5224	0.50	0.87	1.00	0.87	2612.00	4544.88	5224.00	4544.88
60	8097	0.87	0.87	0.00	−0.87	7044.39	7044.39	0	−7044.39
90	7850	1.00	0.00	−1.00	0.00	7850.00	0.00	−7850.00	0.00
120	5499	0.87	−0.87	0.00	0.87	4784.13	−4784.13	0.00	4784.13
150	2626	0.50	−0.87	1.00	−0.87	1313.00	−2284.62	2626.00	−2284.62
					Total:	23603.52	4520.52	0.00	0.00

Thus, $b_1 = \dfrac{23603.52}{3} = 7867.67 \approx 7868,$

$$b_2 = \frac{4520.52}{3} = 1506.84 \approx 1507,$$

$$b_3 = 0, \quad b_4 = 0.$$

Hence, the Fourier series is given by

$$T \approx 7868 \sin\theta + 1507 \sin 2\theta.$$

When $\theta = 75°$, then

$$\begin{aligned} T &= 7868 \sin 75° + 1507 \sin 150° \\ &= 7868 \times 0.9659 + 1507 \times 0.50 \\ &\approx 8353.20. \end{aligned}$$

Example 17.23: The following table gives the variations of a periodic current over a fundamental period of T second

t(sec)	:	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A(amp)	:	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Show that there is direct current part of 0.75 amp. in the variable current and obtain the amplitude of the first harmonic.

Solution: The series is periodic over the interval $(0, T)$ hence the period $2l = T$, that is, $l = T/2$. Thus the current A is given as

$$A = a_0 + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + b_2 \sin \frac{4\pi t}{T}.$$

Here a_0 represents the direct current part and $\sqrt{a_1^2 + b_1^2}$ gives the amplitude of the first harmonic.

To calculate the coefficients, we form the following table:

t	$2\pi t/T$	$\cos(2\pi t/T)$	$\sin(2\pi t/T)$	A	$A\cos(2\pi t/T)$	$A\sin(2\pi t/T)$
0	0	1.00	0.000	1.98	1.98	0.00
$T/6$	$\pi/3$	0.50	0.87	1.30	0.65	1.13
$T/3$	$2\pi/3$	−0.50	0.87	1.05	−0.53	0.91
$T/2$	π	−1.0	0.00	1.30	−1.30	0.00
$2T/3$	$4\pi/3$	−0.5	0.87	−0.88	0.44	0.76
$5T/6$	$5\pi/3$	0.5	0.87	−0.25	−.13	0.22
			Total:	4.5	1.11	3.02

Here, $\Sigma A = 4.5$, $\Sigma A \cos(2\pi t/T) = 1.11$, $\Sigma A \sin(2\pi t/T) = 3.02$. Hence

$$a_0 = \frac{4.5}{6} = 0.75, \quad a_1 = \frac{1.11}{3} = 0.37. \quad b_1 = \frac{3.02}{3} = 1.01.$$

Thus, the direct current part is 0.75 amp. and amplitude of the first harmonic is $\sqrt{(0.37)^2 + (1.01)^2} = 1.07$ amp.

EXERCISE 17.5

1. The following values of y give the displacement of a certain machine part for the rotation x of the flywheel

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y :	1.98	2.15	2.77	−0.22	−0.31	1.43	1.98

Express y in Fourier series upto the third harmonic.

2. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in the form of a Fourier series upto fourth hormonic

x :	0	30°	60°	90°	120°	150°	180°
y :	0	9.2	14.4	17.8	17.3	11.7	0

3. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x	:	0	1	2	3	4	5
y	:	4	8	15	7	6	2

4. The turning moment T on the crank-shaft of a steam engine for the crank angle θ in degrees is recorded as follow. Express T in a series of sines upto the fourth harmonic

θ	:	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
T	:	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

5. A part of a machine has an oscillatory motion. The displacement y at a time t is given below:

t	:	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
y	:	0	0.64	1.13	1.34	0.95	0.00	–0.92	–1.33	–1.17	–0.66	0.0

Find constants in the equation $y = A \sin (10\,\pi t + \alpha_1) + B \sin (20\,\pi t + \alpha_2)$.

ANSWERS

Exercise 17.1 (p. 163)

1. $\dfrac{\pi^2}{12} + \dfrac{\cos x}{1^2} + \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} + \ldots$ 2. $-\dfrac{4}{\pi}\left(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x + \ldots\right)$

3. $-1 + \pi \sin x - \dfrac{1}{2}\cos x + \dfrac{2}{2^2-1}\cos 2x + \dfrac{2}{3^2-1}\cos 3x + \ldots$

4. $-\dfrac{\pi^2}{3} + 4\left[\dfrac{\cos x}{1^2} - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \ldots\right] + 2\left[\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dfrac{\sin 4x}{4} + \ldots\right]$

5. $\dfrac{\pi^2}{3} - 4\left(\dfrac{\cos x}{1^2} - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \dfrac{\cos 4x}{4^2} + \ldots\right)$

7. $\dfrac{\sinh 8}{8} + \sinh 8 \sum_{n=1}^{\infty}\left[\dfrac{16(-1)^n}{64+\pi^2 n^2}\cos\left(\dfrac{n\pi x}{2}\right) + \dfrac{2n\pi(-1)^n}{64+\pi^2 n^2}\sin\left(\dfrac{n\pi x}{2}\right)\right]$

8. $\dfrac{k}{2} + \dfrac{2k}{\pi}\left(\cos\dfrac{\pi x}{2} - \dfrac{1}{3}\cos\dfrac{\pi x}{3} + \dfrac{1}{5}\cos\dfrac{\pi x}{5} - \ldots\right)$

9. $4\left(\dfrac{1}{2} - \dfrac{1}{1.3}\cos 2\pi x - \dfrac{1}{3.5}\cos 4\pi x - \dfrac{1}{5.7}\cos 6\pi x \ldots\right)$

Exercise 17.2 (p. 171)

1. $\dfrac{1}{5} - \dfrac{8}{\pi^4}\left[\dfrac{\pi^2-6}{1^4}\cos \pi x - \dfrac{2^2\pi^2-6}{2^4}\cos 2\pi x + \dfrac{3^2\pi^2-6}{3^4}\cos 3\pi x + \ldots\right]$

2. $1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \ldots$

3. $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$ 4. $\frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right)$

5. $\frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$ 6. $\frac{\pi}{2} + 1 - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$

7. $\frac{8}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} - \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} - \ldots\right]$

8. $1, \frac{4}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \ldots\right]$

9. $\frac{8}{3} + \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\frac{n\pi}{2} - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\cos\frac{n\pi x}{4}$

10. $\frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]$

11. $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1.3}\cos\frac{2\pi x}{l} + \frac{1}{3.5}\cos\frac{4\pi x}{l} + \frac{1}{5.7}\cos\frac{6\pi x}{l} + \ldots\right]$

12. $\frac{1}{\pi} + \frac{1}{\pi}\left[\cos x - \frac{2}{3}\cos 2x - \cos 3x - \frac{2}{15}\cos 4x + \frac{1}{3}\cos 5x - \ldots\right]$

Exercise 17.3 (p. 177)

1. $\frac{-\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

2. $\pi^2 - x^2 = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}$, $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$

$x(\pi^2 - x^2) = 12\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^3}$

3. $f(x) = \frac{1}{4}\pi + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$

$$g(x) = \frac{1}{4}x\pi + \frac{1}{4}\pi^2 \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^3}\sin nx + \frac{(-1)^{n+1}}{n^2}(-\cos nx + (-1)^n)\right]$$

4. $x\cos x + \sin x = \frac{1}{2}\pi \sin x + 2\pi \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1} n \sin(nx)$

5. $f'(x) = \frac{1}{\pi}\left(\frac{-2}{3}\sin x + \frac{6}{5}\sin 3x + \ldots\right) + \frac{1}{\pi}\left(\frac{-2}{3}\cos x + \frac{3\pi}{2}\cos 2x - \ldots\right)$

8. $\frac{\pi^4}{96}$

Exercise 17.4 (p. 183)

1. $f(x) = 1 + \lim_{k\to\infty} \sum_{n=-k}^{k} \frac{i}{n\pi}(1 - (-1)^n)e^{-in\pi x}$

2. $f(x) = e - \lim_{k\to\infty} \sum_{n=-k}^{k} \frac{e^{-in\pi x}}{1 - 2n\pi i}$, $n = 0, \pm 1, \pm 2, \ldots$

3. $f(x) = -2\frac{E}{\pi} \lim_{k\to\infty} \sum_{n=-k}^{k} \frac{1}{(4n^2 - 1)} e^{2n\lambda ix}$.

4. $f(x) = \sum_{n=-\infty}^{\infty} \frac{2}{4 + n^2\pi^2}[\{1 - (-1)^n e^{-2}] e^{\frac{-in\pi x}{2}}$

5. $\left(2n, \frac{4|n|}{\pi(4n^2 - 1)}\right)$

6. $\left(n\pi, \frac{[1 - (-1)^n]}{|n|\pi}\right)$

7. $\left(2n\lambda, \left|\frac{2E}{(4n^2 - 1)\pi}\right|\right)$

8. The frequency spectrum consists of the point (0, 4) and the points $(2n, 2/|n|)$.

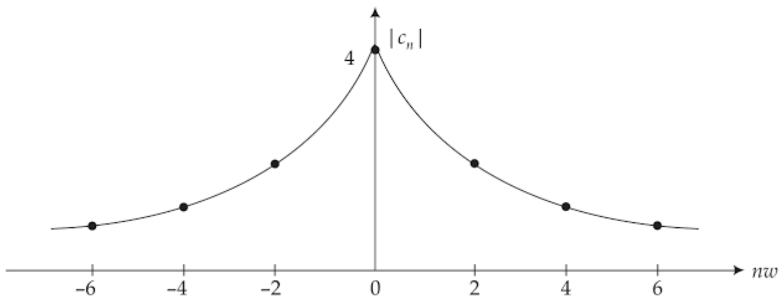


Fig. 17.16

Exercise 17.5 (p. 187)

1. $y \approx 1.3 + 0.92 \cos x + 1.097 \sin x - 0.42 \cos 2x - 0.681 \sin 2x + 0.36 \cos 3x$
2. $y \approx 11.73 - (7.73 \cos 2x + 1.57 \sin 2x) + (-2.83 \cos 4x + .116 \sin 4x)$
3. $y \approx 7 - 2.8 \cos x - 1.5 \cos 2x + 2.7 \cos 3x$
4. $T \approx 7.8 \sin \theta + 1.5 \sin 2\theta - 9.2 \sin 3\theta + 11.6 \sin 4\theta$
5. $A = 1.317, \quad B = -0.1524, \quad \alpha_1 = -0.0083, \quad \alpha_2 = -0.315.$

18 Fourier Integrals and Fourier Transforms

CHAPTER

> "Fourier integrals and Fourier transforms extend the concept of Fourier series to non-periodic functions defined for all x. A non-periodic function which cannot be represented as Fourier series over the entire real line may be represented in an integral form. Fourier transforms are integral transforms similar to Laplace transforms. In fact, 'Fourier analysis', the term including various kinds of Fourier series, integrals and transforms find variety of applications in science and engineering".

18.1 FOURIER INTEGRAL

In the preceding chapter we have seen that if a function $f(x)$ is defined on $-\infty < x < \infty$ and is periodic over an interval $-l < x < l$ (and satisfies the other conditions), then it can be represented by a Fourier series. In many practical problems we come across functions defined on $-\infty < x < \infty$ that are not periodic, e.g. $f(x) = e^{-x^2}$, the graph of which is shown in Fig. 18.1.

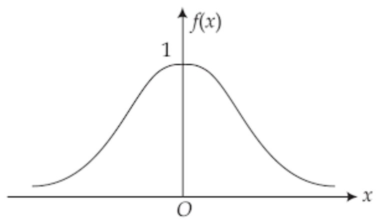


Fig. 18.1

We cannot expand such functions in Fourier series since they are not periodic, however we can consider such functions to be periodic but with an infinite period. *The Fourier integral can be regarded as an extension of the concept of Fourier series to non-periodic (or aperiodic) functions by letting* $l \to \infty$.

Consider any periodic function $f(x)$ of period $2l$ that can be represented by a Fourier series, then

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right), \qquad ...(18.1)$$

where $a_0 = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x)dx$, $a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$ and $b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$.

Substituting the values for a_0, a_n and b_n, (18.1) becomes

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)du + \frac{1}{l}\sum_{n=1}^{\infty}\left[\cos\frac{n\pi x}{l}\int_{-l}^{l} f(u)\cos\frac{n\pi u}{l}du + \sin\frac{n\pi x}{l}\int_{-l}^{l} f(u)\sin\frac{n\pi u}{l}du\right]. \quad ...(18.2)$$

Set $w_n = \frac{n\pi}{l}$ and $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}$, (18.2) becomes

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)du + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w\int_{-l}^{l} f(u)\cos w_n u du + (\sin w_n x)\,\Delta w\int_{-l}^{l} f(u)\sin w_n u du\right]. \quad ...(18.3)$$

The Eq. (18.3) is valid for any fixed finite l, arbitrary large.

We now let $l \to \infty$, and assume $f(x)$ to be absolutely integrable over the interval $(-\infty, \infty)$, that is, $\int_{-\infty}^{\infty} |f(x)|dx$ converges, then the value of the integral $\frac{1}{2l}\int_{-l}^{l} f(u)du$ tends to zero as $l \to \infty$; also $\Delta w = \pi/l \to 0$ and the infinite series in (18.3) becomes an integral from 0 to ∞, which represents $f(x)$ as

$$f(x) = \int_{0}^{\infty} [A(w)\cos wx + B(w)\sin wx]dw, \quad ...(18.4)$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\,du \text{ and } B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu\,du, \quad ...(18.5)$$

are called the *Fourier coefficients* and (18.4) is called the *Fourier integral representation of* $f(x)$.

The sufficient conditions for the validity of (18.4) are

1. *$f(x)$ is piecewise continuous on every interval $[-l, l]$.*
2. *$f(x)$ is absolutely integrable on the real axis, that is, $\int_{-\infty}^{\infty} |f(x)|dx$ converges.*
3. *At every x on the real line $f(x)$ has left and right hand derivatives.*

We state without proof the following convergence theorem for the Fourier integral, called the *Fourier Integral Theorem*.

Theorem 18.1 (Fourier integral theorem): *If $f(x)$ satisfies the conditions 1 to 3 stated above, then the Fourier integral of f converges to $f(x)$ at every point x at which f is continuous, and to the mean value $[f(x + 0) + f(x - 0)]/2$ at every point x at which f is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.*

Example 18.1: Find the fourier integral representation of $f(x) = \begin{cases} 1, & \text{for } -1 \le x \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$ and hence prove that $\int_0^\infty \frac{\sin w}{w} dw = \frac{\pi}{2}$.

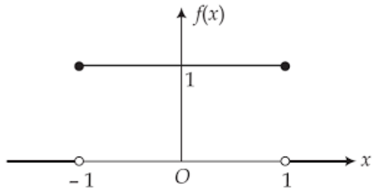


Fig. 18.2

Solution: The graph of $f(x)$ is shown in Fig. 18.2. Clearly $f(x)$ is piecewise smooth and is absolutely integrable over $(-\infty, \infty)$. Thus $f(x)$ has a Fourier integral representation. The Fourier coefficients of $f(x)$ are

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\,du = \frac{1}{\pi}\int_{-1}^{1}\cos wu\,du = \left[\frac{\sin wu}{\pi w}\right]_{-1}^{1} = \frac{2\sin w}{\pi w},$$

and,

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu\,du = \frac{1}{\pi}\int_{-1}^{1}\sin wu\,du = 0.$$

Hence the Fourier integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw.$$

Since $x = \pm 1$ are the points of discontinuity of $f(x)$, thus at $x = \pm 1$

$$\frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw = \frac{1}{2}[f(x+0) + f(x-0)], \text{ which gives}$$

$$\frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw = 1/2, \text{ for } \quad x = \pm 1.$$

$$\text{Thus, } \int_0^\infty \frac{\cos wx \sin w}{w} dw = \begin{cases} \frac{\pi}{2}, & \text{for } -1 < x < 1 \\ \frac{\pi}{4}, & \text{for } x = \pm 1 \\ 0, & \text{for } |x| > 1 \end{cases} \qquad \text{...(18.6)}$$

Set $x = 0$ in (18.6), we have $\int_0^\infty \frac{\sin w}{w} dw = \pi/2.$

Example 18.2: Find the Fourier integral representation of $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

and find the value of the resulting integral when, (a) $x < 0$, (b) $x = 0$, (c) $x > 0$. Also derive that $\int_0^\infty \frac{dw}{1+w^2} = \pi/2.$

Solution: The given function $f(x)$ is piecewise smooth and is absolutely integrable over $(-\infty, \infty)$, since $\int_{-\infty}^{\infty} |f(x)|dx = \int_0^\infty e^{-x}dx = 1$. Thus $f(x)$ has a Fourier integral representation. The Fourier coefficients of $f(x)$ are

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\, du = \frac{1}{\pi}\int_0^\infty e^{-u}\cos wu\, du = \frac{1}{\pi(1+w^2)},$$

and,

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu\, du = \frac{1}{\pi}\int_0^\infty e^{-u}\sin wu\, du = \frac{w}{\pi(1+w^2)}.$$

Thus the Fourier integral repesentation of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}dw$$

At the point of discontinuity, $x = 0$,

$$\frac{1}{\pi}\int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}dw = \frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2}[1+0] = \frac{1}{2}.$$

Thus,
$$\int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}dw = \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases} \qquad \text{...(18.7)}$$

Set $x = 0$ in (18.7), we have $\int_0^\infty \frac{dw}{1+w^2} = \pi/2.$

18.2 FOURIER COSINE AND FOURIER SINE INTEGRALS

For an even or odd function the Fourier integral becomes simpler, analogous to the Fourier series expansion for the even or odd function. When $f(x)$ is an even function, then $f(u)$ sin wu is an odd function of u, so from (18.5) we have $B(w) = 0$ and

$$A(w) = \frac{2}{\pi}\int_0^\infty f(u)\cos wu\,du. \qquad \text{...(18.8)}$$

Thus (18.4) simplifies to

$$f(x) = \int_0^\infty A(w)\cos wx\,dw, \qquad \text{...(18.9)}$$

called the *Fourier cosine integral representation* of $f(x)$.

Similarly, when $f(x)$ is an odd function, then $f(u)$ cos wu is an odd function of u, so from (18.5), we have $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(u)\sin wu\,du. \qquad \text{...(18.10)}$$

Thus (18.4) simplifies to

$$f(x) = \int_0^\infty B(w)\sin wx\,dw, \qquad \text{...(18.11)}$$

called the *Fourier sine integral representation* of $f(x)$.

The convergence result for the integral representations of even and odd functions is as follows.

Theorem 18.2: *If $f(x)$ is (i) piecewise continuous on each interval [0, b], (ii) absolutely integrable on the real axis, and (iii) at every $x \in [0, \infty]$, $f(x)$ has left and right hand derivatives then the Fourier cosine and sine integrals of f converge to $f(x)$ at every point x at which f is continuous, and to the mean value $[f(x + 0) + f(x - 0)]/2$ at every point x at which f is discontinuous.*

Also similar to Fourier cosine and sine series defined on half period $[0, l]$, we can define Fourier cosine and Fourier sine integral representations of functions defined on the real half line $[0, \infty]$ by using respectively even or odd expansion of $f(x)$ to the whole real line.

Example 18.3: Find the Fourier cosine and sine integrals of

$$f(x) = e^{-kx}, \quad x > 0, \quad k > 0.$$

Solution: Clearly $f(x)$ is differentiable and is absolutely integrable over $(0, \infty)$.

To obtain Fourier cosine representation, we have

$$A(w) = \frac{2}{\pi}\int_0^\infty f(u)\cos wu\,du = \frac{2}{\pi}\int_0^\infty e^{-ku}\cos wu\,du.$$

Consider

$$I = \int_0^\infty e^{-ku} \cos wu\, du = \left[\frac{e^{-ku}}{w^2 + k^2} (w \sin wu - k \cos wu) \right]_0^\infty .$$

When u tends to infinity, it becomes zero, and when u tends to zero, it becomes $-k/(w^2 + k^2)$, since $k > 0$. Thus $I = k/(w^2 + k^2)$ and hence

$$A(w) = \frac{2}{\pi} \int_0^\infty e^{-ku} \cos wu\, du = \frac{2k}{\pi(w^2 + k^2)} . \qquad ...(18.12)$$

Thus the Fourier cosine integral $f(x) = \int_0^\infty A(w) \cos wx\, dx$ becomes

$$e^{-kx} = \frac{2k}{\pi} \int_0^\infty \frac{\cos wx}{k^2 + w^2} dx,\ k > 0. \qquad ...(18.13)$$

On the similar lines, to obtain the Fourier sine integral representation of $f(x)$, we have

$$B(w) = \frac{2}{\pi} \int_0^\infty e^{-ku} \sin wu\, du = \frac{2w}{\pi(k^2 + w^2)}, \qquad ...(18.14)$$

and thus

$$f(x) = e^{-kx} = \frac{2}{\pi} \int_0^\infty \frac{w \sin wx}{k^2 + w^2} dw,\ w > 0. \qquad ...(18.15)$$

The integral representations (18.12) and (18.14) are called *Laplace integrals* because $A(w)$ is $2/\pi$ times the Laplace transform of $\cos wx$ and $B(w)$ is $2/\pi$ times the Laplace transform of $\sin wx$.

Example 18.4: Let $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$. Using the Fourier cosine integral representation of f, show that $\int_0^\infty \frac{\sin t}{t} dt = \pi/2$.

Solution: The function $f(x)$ is piecewise smooth and is also absolutely integrable over $(0, \infty)$. To obtain Fourier cosine representation, we have

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u) \cos wu\, du = \frac{2}{\pi} \int_0^1 \cos wu\, du = \frac{2}{\pi} \left[\frac{\sin wu}{w} \right]_0^1 = \frac{2 \sin w}{\pi w} .$$

Thus the Fourier cosine integral representation of $f(x)$ is

$$f(x) = \int_0^\infty A(w)\cos wx\,dw = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wx\,dw.$$

The representation converges to $f(x)$ for every x in $(0, \infty)$ except at the point $x = 1$ which is a point of discontinuity of $f(x)$.

At $x = 1$, the representation converges to

$$\frac{f(x+0)+f(x-0)}{2} = \frac{f(1+0)+f(1-0)}{2} = \frac{1}{2}.$$

Therefore $$\int_0^\infty \frac{\sin w}{w}\cos wx\,dw = \begin{cases} \pi/2, & 0 < x < 1 \\ \pi/4, & x = 1 \\ 0, & x > 1 \end{cases} \qquad \text{...(18.16)}$$

and, hence for $x = 1$, (18.16) gives

$$\int_0^\infty \frac{\sin w\cos w}{w}dw = \frac{\pi}{4}, \text{ or } \int_0^\infty \frac{\sin 2w}{2w}dw = \frac{\pi}{4}.$$

Setting $2w = t$ in it, we obtain $\displaystyle\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}.$

Example 18.5: Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin ax\,dx = e^{-a}$, where a is constant.

Solution: The given integral is Fourier sine integral represention.

Let $$f(x) = \int_0^\infty A(w)\sin wx\,dw, \qquad \text{...(18.17)}$$

where $$A(w) = \frac{2}{\pi}\int_0^\infty f(u)\sin wu\,du. \qquad \text{...(18.18)}$$

Comparing (18.18) with the given equation, we get

$$w = a \quad \text{and} \quad \frac{\pi A(w)}{2} = e^{-a}, \text{ thus } \quad A(w) = \frac{2}{\pi}e^{-w},$$

and hence from (18.17) we have

$$f(x) = \frac{2}{\pi}\int_0^\infty e^{-w}\sin wx\,dw. \qquad \text{...(18.19)}$$

Consider $$I = \int_0^\infty e^{-w} \sin wx\, dw = \left[-\frac{e^{-w}}{1+x^2}(x \cos wx + \sin wx)\right]_0^\infty$$

When w tends to infinity, this becomes zero and at $w = 0$, it is $-x/(1 + x^2)$ and thus $I = x/(1 + x^2)$. Using in (18.19), we get

$$f(x) = \frac{2x}{\pi(1+x^2)}, \quad x > 0.$$

18.3 THE COMPLEX FOURIER INTEGRAL REPRESENTATION

Analogous to the complex form of the Fourier series discussed in Section 17.6, the Fourier integral can also be expressed in the equivalent complex form. This complex form provides the necessary plateform to develop the Fourier transform, (refer Section 18.5), which are highly developed as a methodology like the Laplace transform.

Substituting the expressions for $A(w)$ and $B(w)$ from (18.5) into (18.4), we obtain

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u)\{\cos wu \cos wx + \sin wu \sin wx\}du\right]dw$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u) \cos w(u-x)du\right]dw. \qquad \text{...(18.20)}$$

Inserting, $\cos w(u - x) = \frac{1}{2}(e^{iw(u-x)} + e^{-iw(u-x)})$, it becomes

$$f(x) = \frac{1}{2\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u)\{e^{iw(u-x)} + e^{-iw(u-x)}\}du\right]dw$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{iw(u-x)}du\, dw + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{-iw(u-x)du}dw. \qquad \text{...(18.21)}$$

In the first integral on the right side of (18.21), replace w by $-w$, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^0 \int_{-\infty}^\infty f(u)e^{-iw(u-x)}du\, dw + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(u)e^{-iw(u-x)}du\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)e^{-iw(u-x)}du\, dw. \qquad \text{...(18.22)}$$

This is the *complex Fourier integral representation* of f on the real line.

If we put

$$c(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-iwu}du, \qquad ...(18.23)$$

then the integral (18.22) becomes

$$f(x) = \int_{-\infty}^{\infty} c(w)e^{iwx}dw. \qquad ...(18.24)$$

The $c(w)$ as given in (18.23) is called the *complex Fourier integral coefficient of* f.

Example 18.6: If $f(x) = e^{-a|x|}$ for all real x and with $a > 0$, a positive constant, then find the complex Fourier integral representation of f.

Solution: The function is

$$f(x) = \begin{cases} e^{-ax}, & \text{for } x \geq 0 \\ e^{ax}, & \text{for } x < 0 \end{cases}$$

$a > 0$ being a constant.

Obvious $f(x)$ is piecewise smooth and is absolutely integrable over the interval $(-\infty, \infty)$.

The complex Fourier integral coefficient of f is given by

$$c(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-iwu}du = \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{au}e^{-iwu}du + \int_{0}^{\infty} e^{-au}e^{-iwu}du\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(a-iw)u}du + \int_{0}^{\infty} e^{-(a+iw)u}du\right] = \frac{1}{2\pi}\left[\left[\frac{e^{(a-iw)u}}{a-iw}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+iw)u}}{-(a+iw)}\right]_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{a+iw} + \frac{1}{a-iw}\right) = \frac{a}{\pi(a^2+w^2)}.$$

Thus, the complex Fourier integral representation, $f(x) = \int_{-\infty}^{\infty} c(w)e^{iwx}dw$, becomes

$$e^{-a|x|} = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+w^2}e^{iwx}dw.$$

EXERCISE 18.1

1. Show that $f(x) = 1$, $(0 < x < \infty)$, cannot be represented by a Fourier integral.

Derive the Fourier integral representations of the following functions (Problems **2-5)**. At which points, if any, does the Fourier integral fail to converge to $f(x)$? To what value does the integral converge at those points?

2. $f(x) = \begin{cases} 100, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$ **3.** $f(x) = \begin{cases} bx/a, & |x| \le a \quad a, b > 0 \\ 0, & |x| > a \end{cases}$

4. $f(x) = \begin{cases} (\pi/2)\cos x, & |x| \le \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$ **5.** $f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

In the following problems, find the integral representation as mentioned

6. $f(x) = e^{-2x} + e^{-3x}$, $x > 0$; cosine representation.

7. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$; cosine representation.

8. $f(x) = \begin{cases} \sinh x, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$; sine representation.

In the following problems, find the complex Fourier integral of the function and determine what this integral converges to

9. $f(x) = xe^{|x|}$, for all real x.

10. $f(x) = \begin{cases} \sin \pi x, & |x| \le 5 \\ 0, & |x| > 5 \end{cases}$ **11.** $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi/2 \\ \sin x, & -\pi/2 < x < 0 \\ 0, & |x| > \pi/2 \end{cases}$

12. Define a suitable function $f(x)$ and use the Fourier integral representation to show that

$$\int_0^\infty \frac{\sin ax}{x} dx = \pi/2, \ (a > 0).$$

13. If $\displaystyle\int_0^\infty f(x) \sin ax \, dx = \begin{cases} 1, & 0 < a < 1 \\ 0, & a > 1 \end{cases}$, then find $f(x)$.

14. Using the Fourier integral representation, show that

$$\int_0^\infty \frac{1 - \cos \pi w}{w} \sin(xw) dw = \begin{cases} \pi/2, & 0 < x \le \pi \\ 0, & x > \pi \end{cases}$$

15. Show that $\displaystyle\int_0^\infty \frac{\sin \pi w \sin wx}{1-w^2}\,dw = \begin{cases} \frac{1}{2}\pi \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

18.4 FOURIER TRANSFORM AND ITS PROPERTIES

An *integral transform* is a transformation that produces from a given function, a new function which depends on a different variable and appears in the form of an integral. These transformations are mainly employed as a tool to solve certain initial and boundary value ordinary and partial differential equations arising in many areas of science and engineering. The Laplace transform as discussed in Chapter 13, is one such transform which has wide applications. Fourier transforms are the next other integral transforms which are of vital importance from the applications viewpoint in solving initial and boundary value problems.

We will discuss three transforms: *The Fourier transform, the Fourier cosine transform* and *the Fourier sine transform;* the first being complex and the latter two real. These transforms are obtained from the corresponding Fourier integral representations.

18.4.1 The Fourier Transform

The complex Fourier integral representation of the function $f(x)$ on the real line, refer Eq. (18.22) is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-iwu}du\right]e^{iwx}dw. \qquad ...(18.25)$$

The expression in bracket, a function of w denoted by $F(w)$, is called the *Fourier transform of f*; and since u is a dummy variable, we replace u by x and have

$$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx, \qquad ...(18.26)$$

so that (18.25) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(w)e^{iwx}dw, \qquad ...(18.27)$$

and is called the *inverse Fourier transform of F(w).*

The function $f(x)$ and the associated Fourier transform $F(w)$ are called a *Fourier transform pair.*

Other common notations used for the Fourier transform of $f(x)$ are $\hat{f}(w)$ or, $\mathcal{F}(f(x))$ and the inverse Fourier transform is denoted by $\mathcal{F}^{-1}(f(x))$. Further, the choice of the normalizing factors

$1/\sqrt{2\pi}$ in integrals (18.26) and (18.27) is optional and it is chosen here so, to make the two integrals as symmetric as possible. All that is required for the normalizing factors is that their product be $1/2\pi$. In fact we can write the normalizing factors in the general form as $k/2\pi$ and $1/k$, where k is an arbitrary scale factor.

The sufficient conditions for the existence of the Fourier transform are:

1. *$f(x)$ is piecewise continuous on every finite interval; and*
2. *$f(x)$ is absolutely integrable on the real axis.*

Similarly, for the existence of inverse Fourier transform of $F(w)$, $F(w)$ must be absolutely integrable over $(-\infty, \infty)$, and thus $\lim_{|w|\to\infty} F(w) = 0$.

Example 18.7: Find the Fourier transforms of

(a) $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$ (b) $f(x) = \begin{cases} a, & -l < x < 0 \\ b, & 0 < x < l \quad a, b > 0 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = u(x+1) - u(x-1)$, where $u(x)$ is the unit-step function.

(d) $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$

Solution: (a) By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_0^a k e^{-iwx} dx = \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_0^a = \frac{k}{iw\sqrt{2\pi}} (1 - e^{-iwa}).$$

(b)
$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-l}^{0} a e^{-iwx} dx + \frac{1}{\sqrt{2\pi}} \int_0^l b e^{-iwx} dx$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_{-l}^{0} + \frac{b}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_0^l = \frac{1}{iw\sqrt{2\pi}} [(b-a) + ae^{-iwl} - be^{-iwl}].$$

(c) The graph of $f(x) = u(x+1) - u(x-1) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

is shown in Fig. 18.3.

By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-iwx}}{-iw} \right)_{-1}^{1}$$

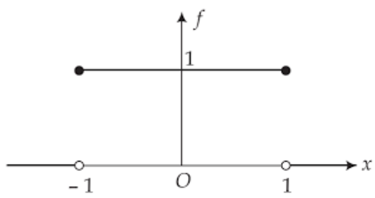


Fig. 18.3

$$= \frac{e^{iw} - e^{-iw}}{\sqrt{2\pi}\, iw} = \sqrt{\frac{2}{\pi}}\, \frac{\sin w}{w}.$$

(d) By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_{-a}^{a}$$

$$= \frac{1}{w\sqrt{2\pi}} \left[\frac{e^{iwa} - e^{-iwa}}{i} \right] = \sqrt{\frac{2}{\pi}}\, \frac{\sin wa}{w}.$$

Example 18.8: Find the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$.

Solution: By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax^2 + iwx)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[-\left(\sqrt{a}x + \frac{iw}{2\sqrt{a}} \right)^2 + \left(\frac{iw}{2\sqrt{a}} \right)^2 \right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-\left[\sqrt{a}\,x + \frac{iw}{2\sqrt{a}} \right]^2} dx = \frac{1}{\sqrt{2\pi a}} e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-t^2} dt, \text{ setting } \sqrt{a}x + \frac{iw}{2\sqrt{a}} = t$$

$$= \frac{1}{\sqrt{2\pi a}} e^{-\frac{w^2}{4a}} . \sqrt{\pi}, \text{ since } \int_{-\infty}^{\infty} e^{-t^2} dt = 2\int_{0}^{\infty} e^{-t^2} dt = \Gamma(1/2) = \sqrt{\pi}.$$

$$= \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}.$$

Example 18.9: Find the Fourier transform of the following functions

(a) $f(x) = e^{-|x|}$ (b) $\delta(x) = \lim_{k \to 0} \frac{1}{k} [u(x) - u(u - k)]$; $u(x)$ being the unit-step function.

Solution: (a) The function is $f(x) = \begin{cases} e^{x} & -\infty < x \le 0 \\ e^{-x} & 0 < x < \infty \end{cases}$

By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(1-iw)x}dx + \int_{0}^{\infty} e^{-(1+iw)x}dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[\frac{e^{(1-iw)x}}{(1-iw)} \right]_{-\infty}^{0} - \left[\frac{e^{-(1+iw)x}}{(1+iw)} \right]_{0}^{\infty} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1-iw)} + \frac{1}{(1+iw)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{1+w^2} = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}.$$

(b) The function $u(x) - u(x-k)$ is

$$u(x) - u(x-k) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x < k \\ 0, & x \ge k \end{cases}$$

Thus $\mathcal{F}(\delta(x)) = \lim_{k \to 0} \left[\frac{1}{k} \mathcal{F}[u(x) - u(x-k)] \right] = \lim_{k \to 0} \left[\frac{1}{k\sqrt{2\pi}} \int_{0}^{k} e^{-iwx}dx \right]$

$$= \frac{1}{\sqrt{2\pi}} \lim_{k \to 0} \left[\frac{1 - e^{-iwk}}{iwk} \right] = \frac{1}{\sqrt{2\pi}}.$$

Remark. A graph of $|F(w)|$ versus w is called the *amplitute spectrum* of $f(x)$.

For example, if $f(x) = u(x+1) - u(x-1)$, then $F(w) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$, refer Example (18.7c). The graph of $\left(w, \sqrt{\frac{2}{\pi}} \left| \frac{\sin w}{w} \right| \right)$ is as shown in Fig. 18.4 for $w \ge 0$.

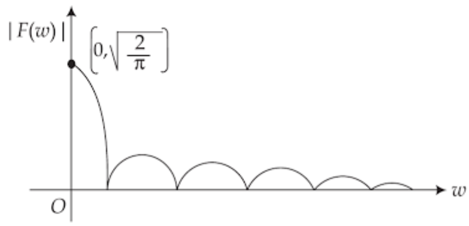


Fig. 18.4

18.4.2 Properties of Fourier Transform

The properties of Fourier transform help to simplify the calculations involving Fourier transform and to obtain some results which are otherwise difficult to obtain.

1. *Linearity:* We state the following result.

Theorem 18.3 (Linearity theorem): *For any functions f(x) and g(x) whose Fourier transforms exist and for any constants a, b*

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x)), \qquad ...(18.28)$$

where $\mathcal{F}(f(x))$ denotes the Fourier transform of $f(x)$.

The proof follows directly from the definition of Fourier transform.

2. *Fourier transform of derivatives:* It is stated as follows:

Theorem 18.4 (Transform of derivatives): *If $f(x)$ is a continuous function of x with $f(x) \to 0$ as $|x| \to \infty$, and $f'(x)$ is absolutely integrable over $(-\infty, \infty)$, then*

(a) $$\mathcal{F}[f'(x)] = iw\mathcal{F}[f(x)], \qquad ...(18.29)$$

(b) $$\mathcal{F}[f^{(n)}(x)] = (iw)^n \mathcal{F}[f(x)], \qquad ...(18.30)$$

and holds for all n such that the derivatives $f^{(r)}(x)$, $r = 1, 2, ..., n$ satisfy the sufficient conditions for the existence of the Fourier transforms.

Proof. (a) By definition, $\mathcal{F}[f'(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f'(x) e^{-iwx} dx.$

Integrating by parts, we obtain

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \left[\left(f(x) e^{-iwx} \right)_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \right]$$

Since $f(x) \to 0$ as $|x| \to \infty$, therefore,

$$\mathcal{F}[f'(x)] = (iw)\mathcal{F}[f(x)]$$

(b) The repeated application of result (a) gives result (b) provided the desired conditions are satisfied at each step.

Example 18.10: Find the Fourier transform of $f(x) = x\, e^{-ax^2}$, $a > 0$.

Solution: We have

$$\mathcal{F}[f(x)] = \mathcal{F}[xe^{-ax^2}] = \mathcal{F}\left[-\frac{1}{2a} (e^{-ax^2})' \right]$$

$$= -\frac{1}{2a} \mathcal{F}[(e^{-ax^2})']$$

$$= -\frac{1}{2a} (iw) \mathcal{F}[e^{-ax^2}], \quad \text{using differentiability}$$

$$= -\frac{iw}{2a} \left(\frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}} \right), \quad \text{refer Example (18.8)}$$

$$= \frac{-iw}{2a\sqrt{2a}} e^{-\frac{w^2}{4a}}.$$

Example 18.11: Show that

(a) $\mathcal{F}[x^n f(x)] = i^n \dfrac{d^n}{dw^n}[F(w)]$ ...(18.31)

(b) $\mathcal{F}[x^m f^{(n)}(x)] = i^{m+n}\dfrac{d^m}{dw^m}[w^n F(w)]$, ...(18.32)

where $F(w) = \mathcal{F}[f(x)]$ is the Fourier transform of $f(x)$.

Solution: (a) By definition of Fourier transform

$$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx.$$

Differentiating it w.r.t. w and using Leibnitz rule to differentiate under the integral sign, we have

$$\frac{d}{dw}[F(w)] = \frac{1}{\sqrt{2\pi}}\frac{d}{dw}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\, f(x)e^{-iwx}dx$$

or,
$$i\frac{d}{dw}[F(w)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{-iwx}dx = \mathcal{F}[xf(x)].$$

The repeated applications of the differentiation w.r.t. w leads to the desired result

$$\mathcal{F}[x^n f(x)] = i^n \frac{d^n}{dw^n}[F(w)].$$

(b) Consider

$$\mathcal{F}[x^m f^{(n)}(x)] = i^m \frac{d^m}{dw^m}\mathcal{F}[f^{(n)}(x)], \quad \text{using (18.31)}$$

$$= i^m \frac{d^m}{dw^m}[(iw)^n F(w)], \quad \text{using (18.30)}$$

$$= i^{m+n}\frac{d^m}{dw^m}[w^n F(w)],$$

provided $f(x)$ and its successive derivatives satisfy the requisite conditions.

Example 18.12: Using the property of the Fourier transform of derivatives, find the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$.

Solution: Clearly $f(x)$ satisfies the requisite conditions of continuity and absolute integrability over the real axis for the existence of Fourier transform.

It is easy to see that $f(x)$ satisfies the differential equation

$$f'(x) + 2a\,xf(x) = 0. \qquad \text{...(18.33)}$$

Taking the Fourier transform of (18.33), we have

$$\mathcal{F}[f'(x)] + 2a\,\mathcal{F}[xf(x)] = 0.$$

It gives $\quad iwF(w) + 2a\,(i\,F'(w)) = 0$

or, $$2a\,F'(w) + wF(w) = 0, \qquad \text{...(18.34)}$$

where $F(w)$ is the Fourier transform of $f(x)$. Rewriting (18.34) as

$$\frac{F'(w)}{F(w)} = -\frac{1}{2a}w.$$

Integrating it w.r.t. w, we get

$$F(w) = A\exp\left[-\frac{w^2}{4a}\right],$$

where A is an arbitrary constant.

To determine A we have $F(0) = A$ and also at $w = 0$,

$$F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \frac{1}{\sqrt{2\pi}}\cdot\frac{\sqrt{\pi}}{\sqrt{a}} = \frac{1}{\sqrt{2a}}.$$

Thus, $$F(w) = \frac{1}{\sqrt{2a}}e^{-w^2/4a}, \quad a > 0.$$

3. Shifting x by x_0 (the time-shifting); Scaling x by a; and Shifting w by w_0 (the frequency-shifting): The result is stated as follow.

Theorem 18.5 (Shifting and scaling): *If $f(x)$ has Fourier transform $F(w)$, then*

(a) $\mathcal{F}[f(x - x_0)] = e^{-iwx_0}\,F(w)$; *shifting on x-axis by* x_0.

(b) $\mathcal{F}[f(ax)] = \dfrac{1}{a}F(w/a)$, $a > 0$; *scalling x by a,*

(c) $\mathcal{F}[e^{iw_0x}f(x)] = F(w - w_0)$; *shifting w by* w_0.

The results follow immediately from the definition of the Fourier transform of $f(x)$.

Example 18.13: Find the Fourier transform of $f(x) = e^{-a(x-5)^2}$, $a > 0$, using shifting property.

Solution: By shifting property, we have

$$\mathcal{F}[e^{-a(x-5)^2}] = e^{-i5w}\mathcal{F}[e^{-ax^2}]$$

$$= e^{-i5w}.\frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}, \quad \text{refer Example (18.8)}$$

$$= \frac{1}{\sqrt{2a}}e^{-\left(\frac{w^2}{4a}+i5w\right)}.$$

Example 18.14: Find the Fourier transform of $f(x) = 4e^{-|x|} - 5e^{-3|x+2|}$.

Solution: Using the linearity property

$$\mathcal{F}[f(x)] = 4\mathcal{F}(e^{-|x|}) - 5\mathcal{F}(e^{-3|x+2|})$$

$$= 4\mathcal{F}(e^{-|x|}) - 5e^{2iw}\mathcal{F}(e^{-|3x|}), \text{ using } x\text{-shifting}$$

$$= 4\mathcal{F}(e^{-|x|}) - \frac{5}{3}e^{2iw}\mathcal{F}(e^{-|x|})_{w\to w/3}, \text{ using scaling}$$

$$= 4.\frac{1}{\sqrt{2\pi}}\frac{2}{1+w^2} - \frac{5}{3}e^{2iw}.\frac{1}{\sqrt{2\pi}}\frac{2}{1+\left(\frac{w}{3}\right)^2}, \text{ refer Example (18.9a)}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{8}{1+w^2} - \frac{30e^{2iw}}{9+w^2}\right],$$

4. Fourier transform of integrals: It is stated as follows.

Theorem 18.6 (Transforms of integrals): *If $\mathcal{F}[f(x)] = F(w)$, then*

$$\mathcal{F}\left[\int_{-\infty}^{x} f(t)\,dt\right] = \frac{1}{iw}F(w), \qquad \text{...(18.35)}$$

provided $F(w)$ satisfies $F(0) = 0$.

Proof. Let $g(x) = \int_{-\infty}^{x} f(t)dt$, then $g'(x) = f(x)$, since $\lim_{x\to-\infty} f(x) = 0$.

Also, $\mathcal{F}[g'(x)] = iw\mathcal{F}[g(x)]$

Substituting for $g(x)$ and $g'(x)$, it becomes

$$\mathcal{F}[f(x)] = iw\mathcal{F}\left[\int_{-\infty}^{x} f(t)\,dt\right],$$

which gives (18.35) .

Example 18.15: Using the transform of integrals, find the Fourier transform of $f(x) = e^{-ax^2}$.

Solution: We have,

$$\mathcal{F}(e^{-ax^2}) = \mathcal{F}\left[-2a\int_{-\infty}^{x} xe^{-ax^2}\,dx\right] = -2a\mathcal{F}\left[\int_{-\infty}^{x} xe^{-ax^2}\,dx\right]$$

$$= -2a\frac{1}{iw}\mathcal{F}(xe^{-ax^2}) = -2a.\frac{1}{iw}\left(\frac{-iw}{2a\sqrt{2a}}e^{-\frac{w^2}{4a}}\right)\text{, refer Example (18.10)}$$

$$= \frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}$$

5. Fourier transform of convolutions: The *convolution* $f*g$ of two functions f and g is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)\,g(x-t)dt$$

We have the following result.

Theorem 18.7 (Convolution theorem): *Let $f(x)$ and $g(x)$ be two piecewise continuous, bounded and absolutely integrable functions on the x-axis, then the Fourier transform of $f*g$, the convolution of f and g, is*

$$\mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g) \qquad \text{...(18.36)}$$

Proof. By definition of Fourier transform

$$\mathcal{F}[(f*g)(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\,g(x-t)\,e^{-iwx}\right)dx\,dt$$

Put $x - t = s$, then $x = (t + s)$, this becomes

$$\mathcal{F}[(f*g)(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,g(s)e^{-iw(t+s)}ds\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}dt \int_{-\infty}^{\infty} g(s)e^{-iws}ds$$

$$= \sqrt{2\pi}\,\mathcal{F}(f(x))\,\mathcal{F}(g(x))$$

We observe that result in case of Fourier transform of convolution is the same as that of Laplace transform of convolution except for the factor $\sqrt{2\pi}$.

Taking inverse Fourier transform of (18.36), we obtain

$$(f*g)(x) = \sqrt{2\pi}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}(f(x))\ \mathcal{F}(g(x))e^{iwx}dw$$

or,
$$(f*g)(x) = \int_{-\infty}^{\infty} F(w)\ G(w)e^{iwx}dw, \qquad \text{...(18.37)}$$

where $F(w) = \mathcal{F}(f(x))$ and $G(w) = \mathcal{F}(g(x))$.

The result (18.37) is particularly useful while solving partial differential equations using Fourier Transforms.

Example 18.16: Find the inverse Fourier transform of $F(w) = \dfrac{1}{(4+w^2)(9+w^2)}$

Solution: Lel $h(x)$ be the inverse Fourier transform, then

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{(4+w^2)(9+w^2)}e^{iwx}\,dw$$

$$= \frac{1}{\sqrt{2\pi}}(f*g)(x), \quad \text{using (18.37)}$$

where $f(x) = \mathcal{F}^{-1}\left(\dfrac{1}{4+w^2}\right)$, $g(x) = \mathcal{F}^{-1}\left(\dfrac{1}{9+w^2}\right)$ and $(f*g)$ is the convolution of f and g.

Using Example (18.9a) and scaling property, we have

$$f(x) = \mathcal{F}^{-1}\left(\frac{1}{4+w^2}\right) = \frac{1}{4}\sqrt{2\pi}\ e^{-2|x|},$$

and,

$$g(x) = \mathcal{F}^{-1}\left(\frac{1}{9+w^2}\right) = \frac{1}{6}\sqrt{2\pi}\,e^{-3|x|}.$$

Hence,

$$h(x) = \frac{1}{\sqrt{2\pi}}(f * g)(x) = \frac{\sqrt{2\pi}}{24} e^{-2|x|} * e^{-3|x|}$$

$$= \frac{\sqrt{2\pi}}{24} \int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|} dt$$

For $x > 0$,

$$\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|} dt = \int_{-\infty}^{0} e^{-2|x-t|} e^{-3|t|} dt + \int_{0}^{x} e^{-2|x-t|} e^{-3|t|} dt + \int_{x}^{\infty} e^{-2|x-t|} e^{-3|t|} dt$$

$$= \int_{-\infty}^{0} e^{-2(x-t)} e^{3t} dt + \int_{0}^{x} e^{-2(x-t)} e^{-3t} dt + \int_{x}^{\infty} e^{-2(t-x)} e^{-3t} dt$$

$$= \frac{6e^{-2x}}{5} - \frac{4e^{-3x}}{5}.$$

Similarly for $x < 0$,

$$\int_{-\infty}^{\infty} e^{-2(x-t)} e^{-3|t|} dt = \frac{6e^{2x}}{5} - \frac{4e^{3x}}{5},$$

and, for $x = 0$

$$\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|} dt = \frac{2}{5}$$

Therefore,

$$h(x) = \begin{cases} \sqrt{2\pi}\left(\frac{1}{20} e^{2x} - \frac{1}{30} e^{3x}\right) & x < 0 \\ \sqrt{2\pi}/60 & x = 0 \\ \sqrt{2\pi}\left(\frac{1}{20} e^{-2x} - \frac{1}{30} e^{-3x}\right) & x > \infty \end{cases}$$

$$= \sqrt{2\pi}\left(\frac{1}{20} e^{-2|x|} - \frac{1}{30} e^{-3|x|}\right), -\infty < x < \infty.$$

The table below gives some functions $f(x)$ and their Fourier transforms $F(\omega)$.

	f(x)	$F(\omega) = \mathcal{F}[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$
1.	$\begin{cases} 1 & \lvert x\rvert < a \\ 0 & \lvert x\rvert > a \end{cases}$, (a > 0), or	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin a\omega}{w}\right)$
2.	$\dfrac{1}{x}$	$\begin{cases} \dfrac{i}{\sqrt{2\pi}}, & \omega > 0 \\ 0 & \omega = 0 \\ \dfrac{-i}{\sqrt{2\pi}}, & \omega < 0 \end{cases}$
3.	$\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$ (0 < a < b)	$\dfrac{1}{\sqrt{2\pi}}\left(\dfrac{e^{-ia\omega} - e^{-ib\omega}}{i\omega}\right)$
4.	$\begin{cases} a - \lvert x\rvert, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{1 - \cos aw}{\omega^2}\right)$
5.	$\dfrac{\sin ax}{x}$, a > 0	$\begin{cases} \sqrt{\pi/2}, & \lvert\omega\rvert < a \\ 0, & \lvert\omega\rvert > a \end{cases}$
6.	$\begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases}$ (a > 0)	$\dfrac{1}{\sqrt{2\pi}} \dfrac{1}{a + i\omega}$
7.	$\begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise} \end{cases}$ (a > 0)	$\dfrac{1}{\sqrt{2\pi}}\left[\dfrac{e^{(a - i\omega)c} - e^{(a - i\omega)b}}{a - i\omega}\right]$
8.	$e^{-a\lvert x\rvert}$,(a > 0)	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{a}{a^2 + \omega^2}\right)$
9.	$xe^{-a\lvert x\rvert}$(a > 0)	$-\sqrt{\dfrac{2}{\pi}} \dfrac{2ia\omega}{(a^2 + \omega^2)^2}$
10.	$\lvert x\rvert e^{-a\lvert x\rvert}$, (a > 0)	$\sqrt{\dfrac{2}{\pi}} \dfrac{(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$
11.	e^{-ax^2}, (a > 0)	$\dfrac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
12.	$\dfrac{1}{a^2 + x^2}$,(a > 0)	$\dfrac{1}{a}\sqrt{\dfrac{\pi}{2}}\, e^{-a\lvert\omega\rvert}$
13.	$\dfrac{x}{a^2 + x^2}$,(a > 0)	$\dfrac{-i}{2a}\sqrt{\dfrac{\pi}{2}}\, we^{-a\lvert\omega\rvert}$

Contd.

14.	$\begin{cases} e^{-x}x^a & x > 0 \\ 0, & x \le 0 \end{cases}$	$\dfrac{\Gamma(a)}{\sqrt{2\pi}(1+iw)^a}$
15.	$\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
16.	$J_0(ax)$, $(a > 0)$	$\sqrt{\dfrac{2}{\pi}}\dfrac{u(a-\lvert w\rvert)}{(a^2-w^2)^{1/2}}$

EXERCISE 18.2

1. Find the Fourier transforms of

(a) $f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$ (b) $f(x) = \begin{cases} e^{iax}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} e^{x}, & |x| < a \\ 0, & \text{otherwise} \end{cases}$ (d) $f(x) = \begin{cases} x, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$

2. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

Hence show that $\displaystyle\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$

3. Find the Fourier transform of

(a) $f(x) = e^{-x^2/2}$ (b) $f(x) = \dfrac{\sin ax}{x}$, a > 0

(c) $f(x) = \begin{cases} x^a e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ (d) $f(x) = \begin{cases} x^2, & |x| < x_0 \\ 0, & \text{otherwise} \end{cases}$

4. Show that if $f(x)$ has a finite jump discontinuity at $x = a$, then

$$\mathcal{F}[f'(x)] = iF(w) - \frac{1}{\sqrt{2\pi}}[f(a+) - f(a-)]e^{-iwa}$$

and hence find the Fourier transform of $f'(x)$ when $f(x) = \begin{cases} x, & 0 \le x < a \\ 0, & \text{otherwise.} \end{cases}$

5. Using Fourier transform solve $y'(x) - 4y(x) = u(x)e^{-4x}$, where $u(x)$ is the unit step function.

6. Using the fact that the Bessel function $J_0(x)$ satisfies the differential equation $xf''(x) + f'(x) + xf(x) = 0$, find the Fourier transform of $J_0(x)$.

7. Use convolution to find the inverse Fourier transform of the functions

(a) $\dfrac{1}{(1+iw)^2}$ (b) $\dfrac{\sin 3w}{w(2+iw)}$

8. Using convolution find the Fourier transform of $f(x) = \begin{cases} xe^{-x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

9. Show that Fourier transform of $f(x) = \begin{cases} 1, & b<x<c \\ 0, & \text{otherwise} \end{cases}$ is $F(w) = \dfrac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$

Using this result find the inverse Fourier transform of

$$\frac{i}{\sqrt{2\pi}} \frac{e^{ib(a-w)} - e^{ic(a-w)}}{(a-w)}$$

10. Evaluate $\int_{-\infty}^{\infty} \delta(x-3)\, u(x-3) e^{-5x} dx$, where $\delta(x)$ is the Dirac-Delta function and $u(x)$ is the unit-step function.

18.5 FOURIER COSINE AND FOURIER SINE TRANSFORMS AND THEIR PROPERTIES

The Fourier cosine and sine transforms can be considered as special cases of the Fourier transform of $f(x)$ when $f(x)$ is even or odd function of x over the real axis.

18.5.1 The Fourier Cosine Transform

Consider $f(x)$ to be a piecewise continuous and absolutely integrable function of x over the real axis and so its Fourier transform $F(w)$ exists, refer (18.26), and is given by

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)[\cos wx - i \sin wx] dx \qquad \text{...(18.38)}$$

Further if $f(x)$ is an even function of x, then $f(x) \cos wx$ is even function of x and $f(x) \sin wx$ is odd function of x and so the right side of (18.38) simplifies to

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx \qquad \text{...(18.39)}$$

is called the *Fourier cosine transform of f(x),* (also denoted by $\mathcal{F}_c$ or $\hat{f}_c$).

The *inverse Fourier cosine transform of $F_c(w)$* corresponding to the inverse Fourier transform (18.27) can be obtained as follow.

Consider
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(w) e^{iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(w)[\cos wx + i \sin wx] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(w) \cos wx \, dx,$$

for we note from (18.39) that $F_c(w)$ and, hence $\mathcal{F}_c(w) \cos wx$ is an even function of w, and $\mathcal{F}_c(w) \sin wx$ is an odd function of w.

The integral denoted by $f(x) = \mathcal{F}_c^{-1}(w)$ and defined as

$$f(x) = \mathcal{F}_c^{-1}(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(w) \cos wx \, dx \qquad \text{...(18.40)}$$

is called *the inverse Fourier cosine transform* of $\mathcal{F}_c(w)$.

18.5.2 The Fourier Sine Transform

Similarly, considering $f(x)$ to be an odd function of x, piecewise continuous and absolutely integrable over the real axis, we arrive at integrals

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin wx \, dx \qquad \text{...(18.41)}$$

(also denoted by $\mathcal{F}_s$, or $\hat{f}_s$), defined as the *Fourier sine transform of* $f(x)$, and its inverse

$$f(x) = \mathcal{F}_s^{-1}(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(w) \sin wx \, dw \qquad \text{...(18.42)}$$

defined as *the inverse Fourier sine transform of $F_s(\omega)$.*

The sufficient conditions for the existence of Fourier cosine and sine transforms are:

1. *f(x) is piecewise continuous on each finite interval [0, l]; and*
2. *f(x) is absolutely integrable on the positive real axis.*

Similarly, as in case of inverse Fourier transform, *for the existence of inverse Fourier cosine and sine transforms, $F_c(w)$ and $F_s(w)$ must be absolute integrable over $(0, \infty)$.*

Remarks: 1. Whenever $f(x)$ is discontinuous, then expression on the left of (18.40) and (18.42) is replaced by $[f(x + 0) + f(x - 0)]/2$ because the Fourier cosine and sine transforms satisfy the same convergence properties as the Fourier transform.

2. We have derived Fourier cosine and sine transforms as special cases of the Fourier transform, when $f(x)$ being even or odd, however, as in case of Fourier cosine and Fourier sine integrals, these two transforms respectively can be defined when $f(x)$ is given on semi-infinite interval say, $0 < x < \infty$, and is extended to the domain $-\infty < x < \infty$ as even or odd function.

Example 18.17: Find Fourier cosine and sine transforms of $f(x) = \begin{cases} 1, & 0 \le x \le a \\ 0, & x > a \end{cases}$

Solution: By definition

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\sin wx}{w} \right)_0^a = \sqrt{\frac{2}{\pi}} \frac{\sin aw}{w},$$

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{-\cos wx}{w} \right)_0^a = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos aw}{w} \right)$$

18.5.3 Properties of Fourier Cosine and Sine Transforms

Like Fourier transform the Fourier cosine and sine transforms also satisfy certain properties which are useful from applications point of view.

1. Linearity: For any two functions $f(x)$ and $g(x)$ whose Fourier cosine and sine transforms exist and for any constants a and b

$$\hat{f}_c [af(x) + bg(x)] = a \hat{f}_c [f(x)] + b \hat{f}_c [g(x)]$$

and,

$$\hat{f}_s [af(x) + bg(x)] = a \hat{f}_s [f(x)] + b \hat{f}_s [g(x)]$$

The proofs follow directly from the definition of Fourier cosine and sine transforms.

2. Shifting w by w_0 and scaling x by a: If $F_c(w)$ and $F_s(w)$ are the Fourier cosine and sine transforms of $f(x)$, then

(a) $\hat{f}_c [\cos (w_0 x) f(x)] = \dfrac{1}{2} [F_c(w + w_0) + F_c(w - w_0)]$

(b) $\hat{f}_c [\sin (w_0 x) f(x)] = \dfrac{1}{2} [F_s(w + w_0) - F_s(w - w_0)]$

(c) $\hat{f}_s [\cos (w_0 x) f(x)] = \dfrac{1}{2} [F_s(w + w_0) - F_s(w - w_0)]$

(d) $\hat{f}_s\,[\sin(w_0 x)f(x)] = \dfrac{1}{2}\,[F_c(w - w_0) - F_c(w + w_0)]$

(e) $\hat{f}_c\,[f(ax)] = \dfrac{1}{a}\,F_c(w/a),\ a > 0$

(f) $\hat{f}_s\,[f(ax)] = \dfrac{1}{a}\,F_s(w/a),\ a > 0.$

These results follow directly from the definitions of the Fourier cosine and sine transforms. For example, to prove (b) we have

$$\begin{aligned}\sin w_0 x \cos wx &= \frac{1}{2}\,[\sin(w_0 + w)x + \sin(w_0 - w)x]\\ &= \frac{1}{2}\,[\sin(w + w_0)x - \sin(w - w_0)x].\end{aligned}$$

Thus,

$$\begin{aligned}\hat{f}_c\,[\sin(w_0x)f(x)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(w_o x)\cos(wx)\,f(x)dx\\ &= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(w + w_0)x\ f(x)\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(w - w_0)x\ f(x)\,dx\right]\\ &= \frac{1}{2}\,[F_s(w + w_0) - F_s(w - w_0)].\end{aligned}$$

To prove (e), we have

$$\hat{f}_c\,[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos wx\,dx = \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\frac{w}{a}x\,dx, = \frac{1}{a}F_c(w/a)$$

3. Fourier cosine and sine transforms of derivatives

Let $f(x)$ and $f'(x)$ be continuous and absolutely integrable on the interval $[0, \infty)$ and $f''(x)$ be piecewise continuous on every subinterval $[0, l]$. Then

(a) $\hat{f}_c\,[f'(x)] = w\,F_s(w) - \sqrt{\dfrac{2}{\pi}}\,f(0)$

(b) $\hat{f}_s\,[f'(x)] = -wF_c(w)$

(c) $\hat{f}_c\,[f''(x)] = -w^2F_c(w) - \sqrt{\dfrac{2}{\pi}}\,f'(0)$

(d) $\hat{f}_s\,[f''(x)] = -w^2F_s(w) + w\sqrt{\dfrac{2}{\pi}}\,f(0)$

Proof. (a) By definition

$$\hat{f}_c\,[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[[f(x)\cos wx]_0^{\infty} + w\int_0^{\infty} f(x)\sin wx\,dx\right]$$

$$= wF_s(w) - \sqrt{\frac{2}{\pi}}\,f(0),\ \text{ assuming that } f(x)\to 0 \text{ as } x\to\infty.$$

The result (b) can be proved on the similar lines as in (a).

To prove (c), by definition

$$\hat{f}_c\,[f''(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f''(x)\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[[f'(x)\cos wx + wf(x)\sin wx]_0^{\infty} - w^2\int_0^{\infty} f(x)\cos wx\,dx\right]$$

$$= -w^2F_c(w) - \sqrt{\frac{2}{\pi}}\,f'(0),\ \text{ assuming } f(x), f'(x)\to 0 \text{ as } x\to\infty.$$

The result (d) can be proved on the similar lines as in (c).

Example 18.18: Find the Fourier cosine and sine transforms of $f(x) = e^{-ax}$, $x \geq 0, a > 0$, by using the Fourier cosine and sine transforms of derivatives.

Solution: Here $f(x) = e^{-ax}$, this gives $f'(x) = -ae^{-ax}$ and $f''(x) = a^2e^{-ax}$.

Thus, $$\hat{f}_c[f''(x)] = \hat{f}_c[a^2e^{-ax}] = a^2\,\hat{f}_c[e^{-ax}] = a^2F_c(w), \quad ...(18.43)$$

where $F_c(w)$ denotes the Fourier cosine transform of $f(x) = e^{-ax}$.

Also, $$\hat{f}_c\,[f''(x)] = -w^2F_c(w) - \sqrt{\frac{2}{\pi}}\,f'(0) = -w^2F_c(w) + a\sqrt{\frac{2}{\pi}}, \quad ...(18.44)$$

since $f'(0) = -a$

From (18.43) and (18.44), we have

$$a^2F_c(w) = -w^2F_c(w) + a\sqrt{\frac{2}{\pi}} \ \text{ or, } \ F_c(w) = \sqrt{\frac{2}{\pi}}\,\frac{a}{w^2+a^2}$$

To find Fourier sine transform, consider

$$\hat{f}_s\,[f''(x)] = a^2\,\hat{f}_s\,[e^{-ax}] = a^2F_s(w), \quad ...(18.45)$$

where $F_s(w)$ denotes the Fourier sine transform of $f(x) = e^{-ax}$.

Also, $$\hat{f}_s\,[f''(x)] = -w^2F_s(w) + w\sqrt{\frac{2}{\pi}}\,f(0) = -w^2F_s(w) + w\sqrt{\frac{2}{\pi}} \quad ...(18.46)$$

From (18.45) and (18.46), we have

$$a^2F_s(w) = -w^2F_s(w) + w\sqrt{\frac{2}{\pi}} \quad \text{or,} \quad F_s(w) = \sqrt{\frac{2}{\pi}}\frac{w}{w^2 + a^2}$$

Remark: While solving second order differential equations using integral transforms when the domain is semi-infinite real line, $(0 < x < \infty)$, we need to choose among the Laplace, Fourier cosine and Fourier sine transforms. The Laplace transform will possibly be the best in case of the initial value problems. In case of boundary-value type, the choice will be between Fourier cosine and sine transforms. To use Fourier cosine transform we need to know $f'(0)$; and to use sine transform we require $f(0)$. Thus, the choice may be made accordingly between the two transforms on the basis of the conditions prescribed. We illustrate this in the example to follow next. However, a few specific applications of Fourier transforms to the solutions of partial differential equations are considered in Section 20.15.

Example 18.19: By applying an integral transform, solve the boundary value problem

$$f''(x) - f(x) = 3e^{-2x}, \quad (0 < x < \infty), \quad f(0) = x_0, \quad f(\infty) \text{ bounded} \qquad ...(18.47)$$

Solution: The domain of definition $0 < x < \infty$ is semi-infinite and the problem is boundary-value problem with $f(0) = x_0$; so the clear choice is to apply Fourier sine transform. Applying Fourier sine transform to (18.47) and using the linearity property, we have

$$\hat{f}_s\,[f''(x)] - \hat{f}_s\,[f(x)] = 3\,\hat{f}_s\,[e^{-2x}]$$

or ,

$$-w^2F_s(w) + w\sqrt{\frac{2}{\pi}}f(0) - F_s(w) = 3\sqrt{\frac{2}{\pi}}\frac{w}{w^2 + 4} \qquad ...(18.48)$$

(refer, Example (18.18) for $a = 2$)

Using $f(0) = x_0$ and solving (18.48) for $F_s(w)$, we have

$$F_s(w) = \sqrt{\frac{2}{\pi}}\frac{wx_0}{w^2 + 1} - 3\sqrt{\frac{2}{\pi}}\frac{w}{(w^2 + 4)(w^2 + 1)}$$

$$= \sqrt{\frac{2}{\pi}}\left[(x_0 - 1)\frac{w}{w^2 + 1} + \frac{w}{w^2 + 4}\right] \qquad ...(18.49)$$

Taking inverse Fourier sine transform of (18.49) and using the linearity property of the inverse transform we have, refer Example (18.18) for $a = 1, 2$,

$$f(x) = (x_0 - 1)e^{-x} + e^{-2x}, \qquad ...(18.50)$$

as the solution of (18.47).

Remark: While using the result of $\hat{f}_s\,[f''(x)]$ to obtain the solution of Eq. (18.47), we have implicitly used that $f(x)$ and $f'(x)$ both tends to zero as x tends to infinity and in fact we can verify that (18.50) satisfies these conditions.

18.6 PARSEVAL IDENTITIES FOR FOURIER TRANSFORMS

The Parseval identities for Fourier transform and Fourier cosine and sine transforms are given by

(a) $\int_{-\infty}^{\infty} f(x)\,\bar{g}(x)dx = \int_{-\infty}^{\infty} F(w)\,\bar{G}(w)dw$ (b) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(w)|^2 dw$

(c) $\int_{0}^{\infty} f(x)\,g(x)dx = \int_{0}^{\infty} F_c(w)G_c(w)dw$ (d) $\int_{0}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} |F_c(w)|^2 dw$

(e) $\int_{0}^{\infty} f(x)g(x)dx = \int_{0}^{\infty} F_s(w)G_s(w)dw$ (f) $\int_{0}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} |F_s(w)|^2 dw,$

where $F(w)$, $F_c(w)$ and $F_s(w)$ are respectively the Fourier transform, Fourier sine and Fourier cosine transforms of $f(x)$ respectively and 'bar' denotes the complex conjugate.

We note that Fourier transform is defined for real and complex functions both, thus to prove (a) consider

$$\int_{-\infty}^{\infty} f(x)\;\bar{g}(x)dx = \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{G}(w)\,e^{-iwx}\,dw\right\}dx$$

$$= \int_{-\infty}^{\infty} \bar{G}(w)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx\right\}dw$$

$$= \int_{-\infty}^{\infty} \bar{G}(w)\,F(w)dw = \int_{-\infty}^{\infty} F(w)\;\bar{G}(w)dw,$$

which is (a).

Put $g(x) = f(x)$ in (a), we obtain

$$\int_{-\infty}^{\infty} f(x)\;\bar{f}(x)dx = \int_{-\infty}^{\infty} F(w)\;\bar{F}(w)dw, \text{ or, } \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(w)|^2 dw,$$

which is (b).

Similarly to prove (c), consider

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{0}^{\infty} f(x)\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} G_c(w)\cos wx\,dw\right\}dx$$

$$= \int_{0}^{\infty} G_c(w)\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos wx\,dx\right\}dw$$

$$= \int_0^\infty G_c(w)F_c(w)dw = \int_0^\infty F_c(w)G_c(w)dw$$

which is (c).

The result (d) follows from (c). Similarly we can prove (e) and (f).

These results are useful in solving certain improper integrals.

Example 18.20: Find the Fourier cosine transform of $f(x) = xe^{-ax}$, $x > 0$, $a > 0$ and then evaluate

$$\int_0^\infty \frac{(a^2 - x^2)(b^2 - x^2)}{(a^2 + x^2)^2 (b^2 + x^2)^2} dx$$

using the Parseval identity for the cosine transforms.

Solution: By definition of the Fourier cosine transform

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos wx\, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \cos wx\, dx,$$

$$= \text{Re.} \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} e^{iwx} dx = \text{Re.} \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-(a-iw)x} dx$$

$$= \text{Re} \sqrt{\frac{2}{\pi}} \left[\left(\frac{xe^{-(a-iw)x}}{-(a-iw)} \right)_0^\infty + \int_0^\infty \frac{e^{-(a-iw)x}}{(a-iw)} dx \right]$$

$$= \text{Re.} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-(a-iw)x}}{(a-iw)} dx = \text{Re.} \sqrt{\frac{2}{\pi}} \left(\frac{e^{-(a-iw)x}}{-(a-iw)^2} \right)_0^\infty$$

$$= \text{Re} \sqrt{\frac{2}{\pi}} \frac{1}{(a-iw)^2} = \text{Re} \sqrt{\frac{2}{\pi}} \frac{(a+iw)^2}{(a^2+w^2)^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2 - w^2}{(a^2 + w^2)^2}$$

Thus, $$F_c(w) = \sqrt{\frac{2}{\pi}} \frac{a^2 - w^2}{(a^2 + w^2)^2}$$

For the Fourier cosine transform the Parseval identity is

$$\int_0^\infty F_c(w)G_c(w)dw = \int_0^\infty f(x)g(x)dx \qquad ...(18.51)$$

Set $f(x) = xe^{-ax}$, $a > 0$ and $g(x) = xe^{-bx}$, $b > 0$ and correspondingly

$F_c(w) = \sqrt{\frac{2}{\pi}} \frac{a^2 - w^2}{(a^2 + w^2)^2}$ and $G_c(w) = \sqrt{\frac{2}{\pi}} \frac{b^2 - w^2}{(b^2 + w^2)^2}$, (18.51) becomes

$$\frac{2}{\pi}\int_0^\infty \frac{(a^2 - w^2)(b^2 - w^2)}{(a^2 + w^2)^2 (b^2 + w^2)^2} dw = \int_0^\infty x^2 e^{-(a+b)x} dx$$

$$= \left[x^2 \frac{e^{-(a+b)x}}{-(a+b)} - (2x)\frac{e^{-(a+b)x}}{(a+b)^2} + 2\frac{e^{-(a+b)x}}{-(a+b)^3} \right]_0^\infty = \frac{2}{(a+b)^3}$$

or,
$$\int_0^\infty \frac{(a^2 - w^2)(b^2 - w^2)}{(a^2 + w^2)^2 (b^2 + w^2)^2} dw = \frac{\pi}{(a+b)^3}$$

Changing w to x, we get the desired integral.

Example 18.21: Using Parseval identities show that

(a) $\displaystyle\int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$ (b) $\displaystyle\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} dx = \frac{\pi}{4a}$

Solution: Consider $f(x) = e^{-ax}$, $a > 0$,

The Fourier cosine and sine transform of $f(x)$, respectively are

$F_c(w) = \sqrt{\frac{2}{\pi}} \frac{a}{w^2 + a^2}$ and $F_s(w) = \sqrt{\frac{2}{\pi}} \frac{w}{w^2 + a^2}$, (refer Example 18.18)

To prove (a) consider the Parseval identity for the Fourier cosine transform

$$\int_0^\infty [F_c(w)]^2 = \int_0^\infty [f(x)]^2 dx$$

Set $f(x) = e^{-ax}$ and $F_c(w) = \sqrt{\frac{2}{\pi}} \frac{a}{w^2 + a^2}$, we have

$$\frac{2}{\pi}\int_0^\infty \frac{a^2}{(w^2 + a^2)^2} dw = \int_0^\infty e^{-2ax} dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}$$

or,
$$\int_0^\infty \frac{1}{(w^2 + a^2)^2} dw = \frac{\pi}{4a^3}$$

Changing w to x we get the desired result.

To prove (b), consider the Parseval's identity for the Fourier sine transform,

$$\int_0^\infty [F_s(w)]^2 = \int_0^\infty [f(x)]^2 dx$$

Set $f(x) = e^{-ax}$ and $F_s(w) = \sqrt{\frac{2}{\pi}} \frac{w}{(w^2 + a^2)}$, we have

$$\frac{2}{\pi}\int_0^\infty \frac{w^2}{(w^2 + a^2)^2} dw = \int_0^\infty e^{-2ax} dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}$$

or,
$$\int_0^\infty \frac{w^2}{(w^2 + a^2)^2} = \frac{\pi}{4a}$$

Changing w to x, we get the desired result.

Example 18.22: Using the Parseval identity prove that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

Solution: Consider $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

The Fourier transform of $f(x)$, refer Example (18.7d) for $a = 1$, is

$$F(w) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w}\right)$$

The Parseval identity for the Fourier transform is

$$\int_{-\infty}^\infty |F(w)|^2 dw = \int_{-\infty}^\infty |f(x)|^2 dx.$$

Substituting for $F(w)$ and $f(x)$, it gives

$$\frac{2}{\pi}\int_{-\infty}^\infty \frac{\sin^2 w}{w^2} dw = \int_{-1}^1 dx = 2 \quad \text{or,} \quad \int_{-\infty}^\infty \frac{\sin^2 w}{w^2} dw = \pi$$

Changing w to x, we get

$$\int_{-\infty}^\infty \frac{\sin^2 x}{x^2} dx = \pi \;\text{ or,}\; \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

18.7 THE FINITE FOURIER COSINE AND SINE TRANSFORMS

The Fourier cosine and sine transforms defined on $[0, \infty]$ are motivated by the respective integral representations of a function. In many applications we are to deal with problems defined on finite intervals, and hence we define *the finite Fourier cosine and sine transforms* using Fourier cosine and sine series instead of integrals.

Finite Fourier cosine transform: Suppose f is piecewise continuous on $[0, \pi]$, then *the finite Fourier cosine transform of f* denoted by $F_c(n)$ is defined as

$$F_c(n) = \int_0^{\pi} f(x) \cos nx \, dx \qquad \text{...(18.52)}$$

for $n = 0, 1, 2, \ldots$

Also the Fourier cosine series representation of $f(x)$ on the interval $[0, \pi]$, refer Eq. (17.30) for $l = \pi$, is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} f(x)dx = \dfrac{1}{\pi} F_c(0)$ and $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\, dx = \dfrac{2}{\pi} F_c(n)$. Thus,

$$f(x) = \frac{1}{\pi} F_c(0) + \frac{2}{\pi} \sum_{n=1}^{\infty} F_c(n) \cos nx \qquad \text{...(18.53)}$$

can be interpreted as the ***inverse finite Fourier cosine transform***.

Finite Fourier sine transform: Suppose f is piecewise continuous on $[0, \pi]$, then *the finite Fourier sine transform* of $f(x)$ denoted by $F_s(n)$ is defined as

$$F_s(n) = \int_0^{\pi} f(x) \sin nx \, dx \qquad \text{...(18.54)}$$

for $n = 1, 2, \ldots$

Also the Fourier sine representation series of $f(x)$ on the interval $[0, \pi]$, refer Eq. (17.31) for $l = \pi$, is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}F_s(n)$. Thus,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n)\sin nx \qquad \text{...(18.55)}$$

can be interpreted as the ***inverse finite Fourier sine transform***.

In case the domain of definition for $f(x)$ is $[0, l]$, then (18.52), (18.53), (18.54) and (18.55) respectively become

$$F_c(n) = \int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad \text{...(18.56)}$$

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{l}, \qquad \text{...(18.57)}$$

$$F_s(n) = \int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx, \qquad \text{...(18.58)}$$

and,

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{l}. \qquad \text{...(18.59)}$$

Example 18.23: If $f(x) = x^2$, $0 \le x \le 4$, find finite Fourier cosine and sine transform of $f(x)$.

Solution: By definition

$$F_c(n) = \int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^{4} x^2\cos\frac{n\pi x}{4}\,dx$$

$$= \left[\frac{4x^2}{n\pi}\sin\frac{n\pi x}{4}\right]_0^4 - \int_0^4 \frac{4}{n\pi}2x\sin\frac{n\pi x}{4}\,dx = -\frac{8}{n\pi}\int_0^4 x\sin\frac{n\pi x}{4}\,dx$$

$$= -\frac{8}{n\pi}\left[\frac{-4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_0^4 = \frac{128}{n^2\pi^2}(-1)^n.$$

Similarly,

$$F_s(n) = \int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^{4} x^2\sin\frac{n\pi x}{4}\,dx$$

$$= \left[\frac{-4}{n\pi}x^2\cos\frac{n\pi x}{4}\right]_0^4 + \int_0^4 2x\frac{4}{n\pi}\cos\frac{n\pi x}{4}\,dx$$

$$= -\frac{64}{n\pi}(-1)^n + \frac{8}{n\pi}\left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4$$

$$= \frac{-(-1)^n 64}{n\pi} + \frac{128}{n^3\pi^3}((-1)^n - 1)$$

Example 18.24: Find $f(x)$, $0 < x < \pi$, if its finite Fourier sine transform is

$$F_s(n) = \frac{1 - \cos n\pi}{n^2\pi^2},\ n = 1, 2, \ldots$$

Solution: By definition

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n) \sin\frac{n\pi x}{l} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2\pi^2}\sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin nx = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^2}$$

EXERCISE 18.3

1. Find the Fourier cosine and Fourier sine transforms of
   (a) $f(x) = e^{-x}$, $x > 0$ (b) $f(x) = xe^{-ax}$
   (c) $f(x) = x^{\alpha-1}$, $0 < \alpha < 1$ (d) $h(x) = \int_0^{\infty} f(x)\, g(x)\, dx$
2. Find the Fourier cosine and Fourier sine transforms of
   (a) $f(x) = \begin{cases}\cos x, & 0 \le x \le a \\ 0, & x > a\end{cases}$ (b) $f(x) = e^{-x}\cos x$, $x > 0$
3. Explain why the following functions have neither Fourier cosine transform nor Fourier sine transform
   (a) $f(x) = 1$ (b) $f(x) = e^x$
4. Find the Fourier sine transform of e^{-ax}, $a > 0$ and prove that

   $$\int_0^{\infty}\frac{x\sin\alpha x}{a^2 + x^2}\,dx = \frac{\pi}{2}e^{-a\alpha}, \quad \alpha > 0.$$

   Hence, obtain the Fourier sine transform of $x/(a^2 + x^2)$.
5. Find the Fourier cosine transform of e^{-ax} and hence evaluate $\int_0^{\infty}\frac{\cos\alpha x}{x^2 + a^2}\,dx$

6. Find the finite Fourier cosine and sine transforms of the following functions defined on $[0, \pi]$

(a) $f(x) = \sin ax, a > 0$ (b) $f(x) = \sinh ax, a > 0$

7. Solve the integral equation $\int_0^\infty f(x) \cos \alpha x \, dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

Hence show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \pi/2.$

8. Solve the integral equation $\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1, & 0 \le \alpha < 1 \\ 2, & 1 \le \alpha < 2 \\ 0, & \alpha \ge 2 \end{cases}$

9. Using Parseval identities for sine and cosine transforms of $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$,

evaluate (a) $\int_0^\infty \frac{(1-\cos x)^2}{x^2} dx$ (b) $\int_0^\infty \frac{\sin^2 x}{x^2} dx.$

10. Solve using a cosine or sine transform the boundary value problem

$f''(x) - 9f(x) = 50e^{-3x}, \quad 0 < x < \infty, \quad f(0) = 0, \quad f(\infty)$ bounded.

ANSWERS

Exercise 18.1 (p. 201)

2. $\frac{100}{\pi} \int_0^\infty \frac{1}{w} [\sin 2w \cos wx + (1 - \cos 2w) \sin wx] dw$ for all x, except at $x = 0$ and $x = 2$; and $f(0) = f(2) = 100$ but Fourier integral converges to the average value 50.

3. $\frac{2b}{aw} \int_0^\infty \frac{\sin wx (\sin wa - wa \cos wa)}{w^2} dw$, for all x except at $x = \pm a$; and $f(a) = b$, $f(-a) = -b$ but Fourier integral converges to the average value $b/2$ and $-b/2$.

4. $\int_0^\infty \frac{\cos \frac{\pi w}{2} \cos wx}{1 - w^2} dw$, for all x.

5. $\frac{1}{\pi}\int_0^\infty \frac{w[\sin wx - \sin w(\pi + x)]}{w^2 - 1}\,dw$, for all x, except at $x = 0$ and $x = \pi$. At $x = 0$ Fourier integral converges to the average value 1/2 and at $x = \pi$, converges to –1/2.

6. $\frac{2}{\pi}\int_0^\infty \left(\frac{2}{4+w^2} + \frac{3}{9+w^2}\right)\cos wx\,dw$

7. $\frac{-2}{\pi}\int_0^\infty \frac{[1+\cos w\pi]}{w^2-1}\cos wx\,dw$

8. $\frac{2}{\pi}\int_0^\infty \frac{[\sin 3w \cosh 3 - w\cos 3w \sinh 3]}{1+w^2}\sin wx\,dw$, except at $x = 3$. At $x = 3$, converges to $\frac{1}{2}\sin h\, 3$, the average of $f(3 + 0)$ and $f(3 - 0)$.

9. $\frac{i}{\pi}\int_{-\infty}^\infty \left[\frac{-2w(1-w^2)^2}{((1-w^2)^2 + 4w^2)^2} - \frac{8w^3}{((1-w^2)^2+4w^2)^2}\right] e^{-iwx}\,dw$; converges to $xe^{|x|}$ for all real x.

10. $i\int_{-\infty}^\infty \left(\frac{\sin 5w}{w^2 - \pi^2}\right) e^{iwx}\,dw$; converges to $\sin \pi x$ for $|x| < 5$ and to zero for $|x| \geq 5$.

11. $\frac{1}{2\pi}\int_{-\infty}^\infty \left[-\frac{\cos(\pi w/2)}{w^2-1} + \frac{i\sin(\pi w/2) - w}{w^2-1} + \frac{1 - w\sin(\pi w/2)}{w^2-1} + i\frac{w}{w^2-1}\cos(\pi w/2)\right]e^{-iwx}\,dw$

converges to $\cos x$, for $0 < x < \pi/2$, to $\sin x$ for $-\pi/2 < x < 0$, to 0 for $|x| > \pi/2$, to 1/2 at $x = 0$, to –1/2 at $x = -\pi/2$ and to 0 at $x = \pi/2$.

13. $f(x) = 2(1 - \cos x)/\pi$, $x > 0$.

Exercise 18.2 (p. 214)

1. (a) $\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{iwa}}{iw}\right)$ (b) $\frac{i}{\sqrt{2\pi}}\left(\frac{1-e^{-i(w-a)}}{a-w}\right)$

(c) $\frac{i}{\sqrt{2\pi}} \frac{(e^{a-iaw} - e^{-a+iaw})}{(1-iw)}$ (d) $\frac{[(1+iaw)e^{-iaw} - 1]}{\sqrt{2\pi}\, w^2}$

2. $-\frac{4}{\sqrt{2\pi}\, w^3}(w \cos w - \sin w)$

3. (a) $e^{-w^2/2}$ (b) $\sqrt{\pi/2}$, $|w| < a$; 0, otherwise

(c) $\frac{\sqrt{a}}{\sqrt{2\pi}\,(1+iw)^2}$ (d) $\sqrt{\frac{2}{\pi}} \frac{[(x_0^2 w^2 - 2)\sin x_0 w + 2x_0 w \cos x_0 w]}{w^3}$

5. $-\frac{1}{8} e^{-4|x|}$.

6. $\sqrt{\frac{2}{\pi}} \frac{1}{(1-w^2)^{1/2}}$, $0 < w^2 < 1$.

7. (a) $u(x) \cdot xe^{-x}$ (b) $\frac{1}{4}[1 - e^{-2(x+3)}]u(x+3) - \frac{1}{4}[1 - e^{-2(x-3)}]u(x-3)$

8. $\frac{1}{\sqrt{2\pi}}(1+iw)^2$

10. $4e^{-15}$.

Exercise 18.3 (p. 227)

1. (a) $F_c(w) = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}$, $F_s(w) = \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2}$

(b) $F_c(w) = \sqrt{\frac{2}{\pi}} \frac{a^2 - w^2}{(a^2+w^2)^2}$ $F_s(w) = \sqrt{\frac{2}{\pi}} \frac{2aw}{(w^2+a^2)^2}$

(c) $F_c(w) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{w\alpha} \cos \frac{\alpha\pi}{2}$, $F_s(w) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{w\alpha} \sin \frac{\alpha\pi}{2}$

(d) $H_c(w) = \int_0^\infty F_c(w)\, G_c(w) dw$, $H_s(w) = \int_0^\infty F_s(w) G_s(w) dw$

2. (a) $F_c(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1-w)}{1-w} + \frac{\sin a(1+w)}{1+w}\right]$,

$F_s(w) = \sqrt{\frac{2}{\pi}} \frac{w}{w^2-1} - \frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1+w)}{1+w} - \frac{\cos a\,(1-w)}{1-w}\right]$

(c) $F_c(w) = \dfrac{1}{\sqrt{2\pi}}\left[\dfrac{1}{1+(1+w)^2} + \dfrac{1}{1+(1-w)^2}\right],$

$F_s(w) = \dfrac{1}{\sqrt{2\pi}}\left[\dfrac{1+w}{1+(1+w^2)} - \dfrac{1-w}{1+(1-w)^2}\right]$

4. $\sqrt{\dfrac{2}{\pi}}\,\dfrac{w}{w^2+a^2},\ \sqrt{\dfrac{\pi}{2}}\,e^{-aw}$

5. $\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{(w^2+a^2)},\ \dfrac{\pi}{2a}\,e^{-\alpha a}$

6. (a) $c_o = (1 - \cos a\pi)/a$

$c_n = \dfrac{1}{2(n-a)}\,[\cos\{(n-a)\pi\} - 1] - \dfrac{1}{2(n+a)}\,[\{(n+a)\pi - 1\}]$

$c_n = 0$, if $n = a$ an integer.

$s_n = \left[\dfrac{\sin(n-a)\pi}{2(n-a)} - \dfrac{\sin(n+a)\pi}{2(n+a)}\right]$, $s_n = \pi/2$, if $n = a$, an integer.

7. $f(x) = \dfrac{2(1-\cos x)}{\pi x^2}$

8. $f(x) = (2 + 2\cos x - 4\cos 2x)/\pi x$.

9. (a) $\pi/2$ (b) $\pi/2$

10. $f(x) = -\dfrac{25}{3}\,xe^{-3x}$

19 Partial Differential Equations

CHAPTER

"Modelling of a process that is distributed in space and time generally leads to a partial differential equation (PDE). The issue of existence and uniqueness of solution in case of a PDE is not straight one as like that an ODE. The solution of a PDE involves arbitrary functions and is generally not unique. Some additional conditions are needed to be specified on the boundary of the region where the solution is defined. Further there are no generally applicable methods to solve non-linear PDE's."

19.1 BASIC CONCEPTS

An equation which involves more than one independepent variable and one or more partial derivatives of the dependent variable with respect to them is called a *partial differential equation* (*PDE*). Mathematical models of physical situations involving two or more independent variables often lead to partial differential equations. A few important partial differential equations are:

1. *One dimensional heat flow equation:* $c^2 u_{xx} = u_t$.
2. *Laplace equation in two dimensions:* $u_{xx} + u_{yy} = 0$.
3. *Laplace equation in three dimensions:* $u_{xx} + u_{yy} + u_{zz} = 0$.
4. *One-dimensional wave equation:* $u_{tt} = c^2 u_{xx}$.
5. *Two-dimensional wave equation:* $u_{tt} = c^2(u_{xx} + u_{yy})$.

The *order* of a *PDE* is the order of the highest order partial derivative of the dependent variable u that occurs in the equation. For example, all the equations given above are of order two. Next, as in ordinary differential equation (*ODE*) the *degree* of a *PDE* is defined as the power of the highest order derivative, occurring in the equation after the equation has been made free of radicals and fractions in its derivatives. The equations given above are of degree one.

A general first order PDE for the function $u(x, y)$ is of the form

$$F(x, y, u, u_x, u_y) = 0, \qquad ...(19.1)$$

where F is an arbitrary function.

In general, a first order PDE for a function $u(x_1, x_2, ..., x_n)$ of the n independent variables $x_1, x_2, ...,$ x_n is of the form

$$F(x_1, x_2, ..., x_n, u, u_{x_1}, u_{x_2}, ..., u_{x_n}) = 0, \quad ...(19.2)$$

where F is an arbitrary function and $u_{x_1} = \dfrac{\partial u}{\partial x_i}$, $i = 1, 2, ..., n$.

Similarly, *a general second order PDE for the function* $u(x,y)$ is of the form

$$G(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \quad ...(19.3)$$

where G is an arbitrary function and $u_x = \dfrac{\partial u}{\partial x}$ and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, etc.

The *PDE* of first and second order are of special importance because they occur frequently in practical problems. Further, most of the second order equations that occur in physical applications are *linear*, that is, in which *the unknown function u(x, y) and its partial derivatives appear linearly.*

The general second order linear *PDE* in unknown variable $z(x, y)$ with constant coefficients may be expressed as

$$a\frac{\partial^2 z}{\partial x^2} + 2h\frac{\partial^2 z}{\partial x \partial y} + b\frac{\partial^2 z}{\partial^2 y} + 2f\frac{\partial z}{\partial x} + 2g\frac{\partial z}{\partial y} + cz = f(x, y). \quad ...(19.4)$$

Using the notations

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \text{ and } \quad t = \frac{\partial^2 z}{\partial y^2}, \text{ it becomes}$$

$$ar + 2hs + bt + 2fp + 2gq + cz = f(x, y).$$

Comparing it with the general equation of the conic

$$ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0,$$

the Eq. (19.4) is classified as

(i) *parabolic*, if $h^2 - ab = 0$,

(ii) *hyperbolic*, if $h^2 - ab > 0$, and

(iii) *elliptic*, if $h^2 - ab < 0$

over the region under consideration.

For example, the one-dimensional heat equation $u_t = c^2 u_{xx}$, (with $u \to z$, $t \to y$), is '*parabolic*'; the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$, (with $u \to z$, $t \to y$), is '*hyperbolic*'; while the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$, (with $u \to z$), is '*elliptic*'. The form of the general solution depends on the type of the equation being solved.

Remark. Since the constant $a, h, b, ...$ may be functions of x and y, thus the discriminant $h^2 - ac$ may be a function of x and y and hence it is also possible that it is zero, positive or negative in different parts of the x, y-plane. Consider the *Tricomi equation* $u_{xx} + xu_{yy} = 0$, which arises in the study of the two-dimensional steady transonic flow past a body such as wing of an aeroplane, here we can check that it is elliptic in the right half plane $x > 0$ and hyperbolic in the left plane $x < 0$. Such an equation is called a *change-of-type equation* with solutions that are qualitatively different in the two half planes.

19.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

In the chapter on *ODE*'s we have seen how ordinary differential equations are formed by the elimination of arbitrary constants. Partial differential equations can be formed by the elimination of arbitrary functions or by the elimination of the arbitrary constants from the relation involving three or more variables.

Elimination of arbitrary function(s): Suppose two arbitrary expressions $u = u(x, y, z)$ and $v = v(x, y, z)$ are connected by the relation

$$F(u, v) = 0, \qquad ...(19.5)$$

where F is an arbitrary function which we need to eliminate.

Assuming z to be dependent variable on x and y. Differentiating (19.5) partially with respect to x we obtain

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0. \qquad ...(19.6)$$

Similarly, differentiating (19.5) partially with respect to y, w e obtain

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0. \qquad ...(19.7)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (19.6) and (19.7), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0,$$

which gives

$$Pp + Qq = R, \qquad ...(19.8)$$

a first order partial differential equation linear in p and q, where

$$P = \frac{\partial(u, v)}{\partial(y, z)}, \quad Q = \frac{\partial(u, v)}{\partial(z, x)} \text{ and } R = \frac{\partial(u, v)}{\partial(x, y)}$$

are functions of x, y and z.

Thus the elimination of an arbitrary function F given by (19.5) leads to a partial differential equation linear in p and q.

Elimination of arbitrary constants: Consider a relation of the type

$$f(x, y, z, a, b) = 0, \qquad ...(19.9)$$

where a and b are two arbitrary constants which we need to eliminate.

Considering z to be dependent on the two independent variables x and y, differentiate (19.9) partially w. r. t. x to obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0. \qquad ...(19.10)$$

Similarly differentiating (19.9) w. r. t. y to obtain

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0. \qquad ...(19.11)$$

The three equations (19.9) to (19.11) consist of two arbitrary constants a and b, and , in general, it will be possible to eliminate a and b from these equations to obtain a relation of the form

$$g(x, y, z, p, q) = 0. \qquad ...(19.12)$$

Thus, the elimination of two arbitrary constants from the relation (19.9) leads to a first order *PDE* (19.12). However, the equation obtained by eliminating the arbitrary constants need not necessarily be linear in p and q, refer Example (19.2). But the *PDE* obtained by eliminating an arbitrary function F from (19.5) is always linear. Even sometimes it may not be possible to eliminate the arbitrary constants using Eqs. (19.9), (19.10) and (19.11). Then we go for the second order partial derivatives and eliminate the arbitrary constants but, the partial differential equation may not be unique.

Further every *ODE* of the nth order may be regarded as derived from a solution containing n arbitrary constants. It might be supposed that every *PDE* of the nth order was similarly derivable from a solution containing n arbitrary functions. However, this is not true in general. The elimination of n arbitrary functions sometimes leads to a *PDE* of still higher order and that too may not be unique.

Example 19.1: Derive a *PDE* by eliminating the arbitrary constants a and c from the equation

$$x^2 + y^2 + (z - c)^2 = a^2 \qquad ...(19.13)$$

which represents the set of all spheres with center along the z-axis.

Solution: Differentiating Eq. (19.13) partially with respect to x and y to obtain respectively

$$x + p(z - c) = 0, \text{ and } y + q(z - c) = 0.$$

Eliminating c from these two equations we obtain

$$yp - xq = 0,$$

a partial differential equation linear in p and q.

Example 19.2: Eliminate the arbitrary constants a and b from the equation

$$(x - a)^2 + (y - b)^2 + z^2 = 1 \qquad ...(19.14)$$

which represents a family of spheres with unit radius and center in the *XOY* plane.

Solution: Differentiate Eq. (19.14) partially with respect to x and y to obtain respectively

$$(x - a) + zp = 0, \text{ and } (y - b) + zq = 0.$$

Substituting for $(x - a)$ and $(y - b)$ from these in (19.14) we obtain

$$(1 + p^2 + q^2)z^2 = 1,$$

a non-linear partial differential equation in p and q.

Example 19.3: Eliminate the arbitrary constants a and b from the equation

$$z = ae^{by} \cos bx. \qquad ...(19.15)$$

Solution: Differentiating Eq. (19.15) partially with respect to x and y to obtain respectively

$$z_x = -abe^{by} \sin bx \qquad ...(19.16)$$

and,

$$z_y = abe^{by} \cos bx. \qquad ...(19.17)$$

It is not easy to eliminate a and b from the equations obtained.

Differentiating Eq. (19.16) again w.r.t. x and Eq. (19.17) w.r.t. y to obtain respectively

$$z_{xx} = -ab^2e^{by} \cos bx, \text{ and } z_{yy} = ab^2e^{by} \cos bx, \text{ which give}$$

$$z_{xx} + z_{yy} = 0 \qquad ...(19.18)$$

a second order partial differential equation.

Remark. We note that Eq. (19.18) obtained above is not necessarily unique. Since, from (19.16) we obtain,

$$z_{xy} = -ab^2e^{by} \sin bx = bz_x$$

and,

$$z_{xx} = -ab^2e^{by} \cos bx = -bz_y.$$

Eliminating b from these equations we obtain, $z_x z_{xx} + z_y z_{xy} = 0$, another partial differential equation.

Example 19.4: Form the *PDE* by eliminating the arbitrary function from the relation

$$f(x^2 + y^2, \ z - xy) = 0. \qquad ...(19.19)$$

Solution: Differentiating Eq. (19.19) partially w.r.t. x and y to obtain respectively

$$2xf_u + (p - y)f_v = 0, \text{ and } 2yf_u + (q - x)f_v = 0,$$

where $u = x^2 + y^2, v = z - xy$ and $f_u = \partial f/\partial u$, etc.

Eliminating f_u and f_v from these equations we obtain

$$\begin{vmatrix} 2x & p - y \\ 2y & q - x \end{vmatrix} = 0, \quad \text{or} \quad yp - xq = y^2 - x^2,$$

a partial differential equation linear in p and q.

Example 19.5: Form the *PDE* by eliminating the arbitrary functions from

$$z = yf(x) + xg(y). \qquad ...(19.20)$$

Solution: Differentiating (19.20) partially with respect to x and y to obtain respectively

$$z_x = yf'(x) + g(y) \text{ and } z_y = f(x) + xg'(y).$$

Differentiating partially again to obtain

$$z_{xy} = f'(x) + g'(y).$$

Consider $xz_x + yz_y = xy[f'(x) + g'(y)] + [xg(y) + yf(x)]$

or, $xz_x + yz_y = xyz_{xy} + z,$

a partial differential equation of second order.

Example 19.6: Form the *PDE* by eliminating the arbitrary functions from

$$z = \frac{1}{y}[f(y - ax) + F(y + ax)]. \quad ...(19.21)$$

Solution: Rewriting (19.21) as $zy = f(y - ax) + F(y + ax)$.

Differentiating it partially with respect to x and y to obtain respectively

$$yz_x = -af' + aF' \text{ and } z + yz_y = f' + F'.$$

Differentiating these again to obtain

$$yz_{xx} = a^2f'' + a^2F'' \text{ and } z_y + z_y + yz_{yy} = f'' + F'',$$

Eliminating f'' and F'' from these two we obtain

$$a^2(2z_y + yz_{yy}) = yz_{xx},$$

a second order partial differential equation.

EXERCISE 19.1

Form the *PDE* by eliminating the arbitrary constants from

1. $z = axy + b$
2. $z = (x + a)(y + b)$
3. $ax^2 + by^2 + z^2 = 1$
4. $z = (x - a)^2 + (y - b)^2$
5. $2z = (ax + y)^2 + b$
6. $z = ae^{-pt} \cos qx \sin ry, \quad p^2 + q^2 = r^2$
7. Find the partial differential equation of all planes which are at a constant distance 'd' from the origin.
8. Find the partial differential equation by eliminating the arbitrary constants from the equation $x^2 + y^2 = (z - c)^2 \tan^2 \alpha$ which represents the set of all right circular cones whose axes coincide with the z-axis.

   Form the partial differential equation by eliminating the arbitrary function(s) from the following equations, (**9** -**15**)
9. $z = f(x^2 + y^2)$
10. $f(x^2 + y^2, x^2 - z^2) = 0$
11. $xyz = f(x + y + z)$
12. $z = f(x + ct) + g(x - ct)$
13. $z = f(x) + e^y g(x)$
14. $z = f\left(\frac{y}{x}\right) + \phi(xy)$
15. $z = f(x \cos \alpha + y \sin \alpha - at) + F(x \cos \alpha + y \sin \alpha + at)$
16. Verify that $f(x^2 - z^2, x^3 - y^3) = 0$ is a solution of the partial differential equation $z(y^2z_x + x^2z_y) = xy^2$.

19.3 TYPES OF SOLUTION OF A PDE

*A **'solution'** of a PDE in some domain D of the independent variables x and y, is a function that has all partial derivatives appearing in the equation and satisfies the equation everywhere in D.*

In the preceding section we have seen that a relation of the type

$$f(x, y, z, a, b) = 0 \quad \text{...(19.22)}$$

leads to, in general, a partial differential equation

$$g(x, y, z, p, q) = 0 \quad \text{...(19.23)}$$

of the first order.

*Any such relation of the form (19.22) which contains two arbitrary constants and is a solution of a partial differential equation (19.23), of the first order is said to be a **'complete solution'** or, a **'complete integral'** of that partial differential equation.*

Such a solution is also called *'integral surface'* of the Eq. (19.23).

Next, a relation of the type

$$F(u, v) = 0 \quad \text{...(19.24)}$$

*involving an arbitrary function F connecting two known functions $u = u(x, y, z)$, $v = v(x, y, z)$ and satisfying a first-order partial differential equation of the type (19.23) is called a **'general integral'** or, **'general surface'** of that partial differential equation.*

It appears that in some sense, a general integral should provide a much broader set of solutions as compared to a complete integral, but, we shall see later that it is possible to find a general integral of a partial differential equation once its complete integral is known.

*The solution obtained by determining the arbitrary constants in the complete integral or the arbitrary function in the general integral by using the given conditions, is called a **'particular integral'** or a **'particular solution'** of the partial differential equations.*

*The envelope of the family of surfaces $f(x, y, z, a, b) = 0$ with parameters a and b, if it exists, is called a **'singular integral'** or, **'singular solution'** of the partial differential equation.*

Here we would like to recall that the equation of the envelope of the two-parameter family $f(x, y, z, a, b) = 0$ is obtained by eliminating a and b from the equations $f = 0$, $\partial f/\partial a = 0$ and $\partial f/\partial b = 0$.

The singular integral is different from the particular integral in the sense that it cannot be obtained from the complete integral by assigning some particular values to the arbitrary constants a and b.

We must note that every first order partial differential equation does not possess solution. For example, the non-linear equation $p^2 + q^2 = -1$ is not satisfied by any real function $z = z(x, y)$.

19.4 THE LAGRANGE'S EQUATION: LINEAR PDE OF THE FIRST ORDER

The linear first order partial differential equation of the form

$$Pp + Qq = R, \quad \text{...(19.25)}$$

where P, Q and R are given functions of x, y and z only, is called the 'Lagrange's equation' in two independent variables x and y.

The generalization of Eq. (19.25) to n independent variables is the equation of the form

$$X_1 p_1 + X_2 p_2 + \dots + X_n p_n = X, \qquad \text{...(19.26)}$$

where X_1, X_2 ..., X_n and X are given functions of n independent variables x_1, x_2, ..., x_n and a dependent variable z and $p_i = \dfrac{\partial z}{\partial x_i}$, $i = 1, 2, ..., n$.

Remark. It should be clearly observed in this connection that the term '*linear*' means that p and q appear in Eq. (19.25) in the linear form, but P, Q, R may be any functions of x, y and z, which is in contrast to the situation in case of ordinary differential equation where the dependent variable must also appear linearly. For example the equation $xp - yq = z^2$ is linear, while the equation $x\,(dz/dx) = z^2$ is not a linear one.

19.4.1 General Solution of the Lagrange's Equation

We have the following result:

Theorem 19.1: *The general solution of the linear partial differential equation*

$$Pp + Qq = R \qquad \text{...(19.27)}$$

is

$$F(u, v) = 0, \qquad \text{...(19.28)}$$

where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent solutions of the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \qquad \text{...(19.29)}$$

Proof: Since $u(x, y, z) = c_1$ is a solution of Eqs. (19.29), we have

$$u_x dx + u_y dy + u_z dz = 0, \qquad \text{...(19.30)}$$

and also from (19.29), we have

$$dx = kP, \quad dy = kQ, \quad dz = kR, \qquad \text{...(19.31)}$$

where $k \neq 0$ is an arbitrary constant. Thus, from (19.30) and (19.31), we obtain

$$Pu_x + Qu_y + Ru_z = 0. \qquad \text{...(19.32)}$$

Similarly, since $v(x, y, z) = c_2$ is also a solution of Eqs. (19.29), we obtain

$$Pv_x + Qv_y + Rv_z = 0. \qquad \text{...(19.33)}$$

Solving Eqs. (19.32) and (19.33) simultaneously for P, Q and R, we obtain

$$\frac{P}{\partial(u, v)/\partial(y, z)} = \frac{Q}{\partial(u, v)/\partial(z, x)} = \frac{R}{\partial(u, v)/\partial(x, y)}. \qquad \text{...(19.34)}$$

Also elimination of arbitrary function F from the relation $F(u, v) = 0$ leads to the linear partial differential equation, refer Section (19.2),

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}. \qquad ...(19.35)$$

Setting each term in (19.34) equal to $\frac{1}{c} \neq 0$, an arbitrary constant, and substituting

$$\frac{\partial(u,v)}{\partial(y,z)} = cP, \quad \frac{\partial(u,v)}{\partial(z,x)} = cQ \quad \text{and} \quad \frac{\partial(u,v)}{\partial(x,y)} = cR$$

in (19.35) and cancelling c from both sides, we obtain $Pp + Qq = R$.

Thus $F(u, v) = 0$, where $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent solutions of the auxiliary Eq. (19.29), is the general solution of the Lagrange's Eq. (19.27).

This result can be easily extended to the case of n independent variables stated as follow:

Theorem 19.2: *If $u(x_1, x_2, ..., x_n, z) = c_1$, $i = 1, 2, ..., n$ are n independent solutions of the auxiliary equations*

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2} = ... = \frac{dx_n}{P_n}=\frac{dz}{R},$$

then the relation

$$\phi(u_1, u_2, ..., u_n) = 0,$$

where ϕ is an arbitrary function, is a general solution of the linear partial differential equation

$$P_1p_1 + P_2p_2 + ... + P_np_n = R,$$

where $$p_i = \frac{\partial z}{\partial x_i}, \quad i = 1, 2, ... n.$$

To solve the linear partial differential equation $Pp + Qq = R$, we need to take the following steps:

1. Write the auxiliary equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

2. Find two independent solution of the auxiliary equations. To find these if it is possible to take any pair, say $\frac{dx}{P} = \frac{dy}{Q}$, when the third variable, in this case z, is absent, then we solve it by applying the method of ordinary differential equation and obtain an integral say $u(x, y) = c$. The second integral can be obtained by selecting another pair, if possible. Even we can use the first integral $u(x, y) = c$ while obtaining the second integral.

Sometimes we can find the multipliers l, m, n which are not necessarily constants such that $lP + mQ + nR = 0$, which implies $ldx + mdy + ndz = 0$. This can be integrated if the expression $ldx + mdy + ndz$ is an exact differential of some function. Also we can try for another set of multipliers λ, μ, ν, with this property to find the second integral.

In case $lP + mQ + nR \neq 0$, but $ldx + mdy + ndz = d(lP + mQ + nR)$, then integrating this we can find an integral surface to the auxiliary equations.

3. After obtaining the two independent solutions $u = c_1$ and $v = c_2$ of the auxiliary equations, the general solution of the Lagrange's equation is then of the form $F(u, v) = 0$, or $u = \phi(v)$, or $v = \psi(u)$, where F, ϕ or ψ are arbitrary functions.

Example 19.7: Find the general solution of the partial differential equation

$$x^2p + y^2q = (x + y)z.$$

Solution: The auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}. \quad ...(19.36)$$

From the first pair of terms in (19.36), $\frac{dx}{x^2} = \frac{dy}{y^2}$ we get the integral

$$x^{-1} - y^{-1} = c_1$$

or,

$$\frac{y - x}{xy} = c_1. \quad ...(19.37)$$

Also from Eq. (19.36) we have

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z},$$

which gives

$$\frac{dx - dy}{x - y} = \frac{dz}{z}. \quad ...(19.38)$$

Integrating (19.38), we obtain the integral

$$\frac{x - y}{z} = c_2. \quad ...(19.39)$$

Therefore, the general solution of the given equation from (19.37) and (19.39) is

$$F\left(\frac{y-x}{xy}, \frac{x-y}{z}\right) = 0,$$

where F is an arbitrary function.

Example 19.8: Find the general solution of the partial differential equation

$$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$$

Solution: The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}. \quad ...(19.40)$$

From (19.40), we have

$$\frac{dx - dy}{(x^2 - y^2) + z(x - y)} = \frac{dy - dz}{(y^2 - z^2) + x(y - z)}$$

or,
$$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)}. \qquad ...(19.41)$$

Cancelling the factor $(x + y + z)$ from the denominator in (19.41) to obtain

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}. \qquad ...(19.42)$$

Integrating (19.42), we get the integral

$$\frac{x - y}{y - z} = c_1. \qquad ...(19.43)$$

Similarly from Eq. (19.40), we obtain

$$\frac{dx - dz}{x - z} = \frac{dy - dx}{y - x},$$

which gives the integral

$$\frac{x - z}{y - x} = c_2. \qquad ...(19.44)$$

Hence the general solution of the given equation is

$$F\left(\frac{x - y}{y - z}, \frac{x - z}{y - x}\right) = 0,$$

where F is an arbitrary function.

Remark. Another independent solution from the auxiliary equation (19.40) can be obtained as follows.

Each of the equation in (19.40) is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz}.$$

Also the equations are equal to

$$\frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}.$$

Equating the above two expressions and cancelling the common factor $x^2 + y^2 + z^2 - yz - zx - xy$ on both sides in denominator, we obtain

$$\frac{x\,dx + y\,dy + z\,dz}{(x + y + z)} = dx + dy + dz$$

or,
$$x\,dx + y\,dy + z\,dz = (x + y + z)d(x + y + z).$$

Integrating, we obtain

$$x^2 + y^2 + z^2 = (x + y + z)^2 + c_2'$$

or,
$$xy + yz + zx = c_2'. \qquad ...(19.45)$$

Hence the general solution using (19.43) and (19.45) can also be expressed as

$$\frac{x - y}{y - z} = f(xy + yz + zx),$$

where f is an arbitrary function.

Example 19.9: Find the general solution of the partial differential equation

$$p - q = \ln (x + y)$$

Solution: The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\ln (x + y)}. \qquad ...(19.46)$$

Taking the first pair of terms in (19.46), $dx = - dy$, the integral is

$$x + y = c_1. \qquad ...(19.47)$$

Next, consider $dx = \dfrac{dz}{\ln (x + y)}$. Using (19.47), we obtain $(\ln c_1)dx = dz$, which gives the integral

$$z - (\ln c_1)x = c_2,$$

or,
$$z - x \ln (x + y) = c_2. \qquad ...(19.48)$$

Hence, the general solution of the given equation using (19.47) and (19.48) is

$$z = x \ln (x + y) + \phi(x + y),$$

where ϕ is an arbitrary function.

Example 19.10: Find the general solution of the partial differential equation

$$px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3).$$

Solution: Rewriting the equation in the standard form, we get

$$x(z - 2y^2)p + y(z - y^2 - 2x^3)q = z(z - y^2 -2x^3). \qquad ... (19.49)$$

The auxiliary equations are

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)}. \qquad ...(19.50)$$

From the last pair of terms in (19.50) we obtain $dy/y = dz/z$, which gives the integral

$$y/z = c_1. \qquad ...(19.51)$$

Next, consider the first and the last terms in (19.50), we have

$$\frac{dx}{x(z - 2y^2)} = \frac{dz}{z(z - y^2 - 2x^3)} \qquad ...(19.52)$$

Using (19.51) in (19.52), we obtain

$$\frac{dx}{x(z-2c_1^2z^2)} = \frac{dz}{z(z-c_1^2z^2-2x^3)}$$

or,
$$\frac{dx}{x} = \frac{(1-2c_1^2z)dz}{(z-c_1^2z^2-2x^3)}. \qquad \text{...(19.53)}$$

Set $z - c_1^2z^2 = t$, which gives $(1 - 2c_1^2z)dz = dt$ in (19.53) , to obtain

$$\frac{dx}{x} = \frac{dt}{t-2x^3},$$

or,
$$\frac{dt}{dx} - \frac{1}{x}t = -2x^2, \qquad \text{...(19.54)}$$

a linear differential equation in t of order one and degree one. The solution of Eq. (19.54) is

$$\frac{t}{x} + x^2 = c_2. \qquad \text{...(19.55)}$$

Substituting for t and then setting $c_1 = \frac{y}{z}$, (19.55) gives

$$\frac{z-y^2}{x} + x^2 = c_2. \qquad \text{...(19.56)}$$

Hence the general solution of the given equation is

$$F\left(\frac{y}{z}, \frac{z-y^2}{x} + x^2\right) = 0,$$

where F is an arbitrary function.

Example 19.11: Find the general solution of the partial differential equation

$$p\cos(x+y) + q\sin(x+y) = z.$$

Solution: The auxiliary equations are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}. \qquad \text{...(19.57)}$$

These imply

$$\frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dz}{z}. \qquad \text{...(19.58)}$$

Setting $(x + y) = t$ in (19.58), we obtain

$$\frac{dt}{\cos t + \sin t} = \frac{dz}{z}$$

Solution: The auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0}. \qquad \text{...(19.63)}$$

From Eqs. (19.63), we obtain

$du = 0, \quad dx + dy + dz = 0, \quad xdx + ydy + zdz = 0.$

Integrating we obtain

$$u = c_1, \quad x + y + z = c_2, \quad x^2 + y^2 + z^2 = c_3.$$

Hence the general solution of the given partial differential equation is

$$u = f(x + y + z,\ x^2 + y^2 + z^2),$$

where f is an arbitrary function.

19.4.2 Particular Integral Passing Through a Given Curve

So far we have found that the general solution of the linear partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$, where $u\,(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent solutions of the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Now we explore how such a general solution may be used to find a particular integral surface which passes through a curve whose parametric equations are $x = x(t),\ y = y(t),\ z = z(t)$, where t is a parameter.

Since the curve lies on the integral surface so we must have

$$u[x(t), y(t), z(t)] = c_1, \quad v[x(t), y(t), z(t)] = c_2.$$

We eliminate t from these two equations to obtain a relation between c_1 and c_2. Substituting $c_1 = u(x, y, z),\ c_2 = v(x, y, z)$ in the relation obtained we arrive at the requisite particular integral.

Example 19.13: Find the integral surface of the linear partial differential equation $yp + xq + 1 = z$ which passes through the curve $z = x^2 + y + 1, \quad y = 2x.$

Solution: First we find the general solution of the equation $yp + xq + 1 = z$.

The auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}. \qquad \text{...(19.64)}$$

From the first pair of terms in (19.64) we obtain

$$y^2 - x^2 = c_1. \qquad \text{...(19.65)}$$

Also from (19.64), $\frac{dx+dy}{x+y} = \frac{dz}{z-1}$, which gives the integral surface

$$\frac{z-1}{x+y} = c_2. \qquad \text{...(19.66)}$$

The parametric equations of the given curve $z = x^2 + y + 1,\quad y = 2x$ are, $x = t$ $y = 2t$, and $z = (t + 1)^2$.

Substituting for x, y and z in (19.65) and (19.66), we obtain $3t^2 = c_1$ and $t + 2 = 3c_2$.

Eliminating t from these we obtain $3(3c_2 - 2)^2 = c_1$

Substituting for c_1 and c_2 respectively from (19.65) and (19.66) we obtain

$$3\left[3\left(\frac{z-1}{x+y}\right)-2\right]^2 = (y^2 - x^2),$$

as the desired particular integral of the given equation.

Example 19.14: Find the integral surface of the linear partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0, z = 1$.

Solution: First we find the general solution of the partial differential equation

$$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z.$$

The auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}. \qquad ...(19.67)$$

Each term in (19.67) is equal to $(dx/x + dy/y + dz/z)/0$, which implies

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0. \qquad ...(19.68)$$

Integrating (19.68), we obtain

$$xyz = c_1. \qquad ...(19.69)$$

Also each term in (19.67) is equal to $\dfrac{(xdx + ydy - dz)}{0}$, which gives

$$x\,dx + y\,dy - dz = 0. \qquad ...(19.70)$$

Integrating (19.70), we obtain

$$x^2 + y^2 - 2z = c_2. \qquad ...(19.71)$$

The parametric equation of the given straight line $x + y = 0$, $z = 1$ are, $x = t$, $y = -t$ and $z = 1$. Substituting these values in (19.69) and (19.71), we obtain $-t^2 = c_1$ and $2(t^2 - 1) = c_2$.

Eliminating t from these equations, we obtain

$$2c_1 + c_2 + 2 = 0. \qquad ...(19.72)$$

Substituting for $c_1 = xyz$ and $c_2 = x^2 + y^2 - 2z$ in (19.72), we obtain

$$x^2 + y^2 + 2xyz - 2z + 2 = 0$$

as the desired integral surface of the given equation.

EXERCISE 19.2

Find the general solution of the following partial differential equations:

1. $xp + yq = x$
2. $(y + z)p + (x + z)q = x + y$
3. $2yzp + zxq = 3xy$
4. $xy^2p + y^3q = (zxy^2 - 4x^3)$
5. $2xzp + 2yzq + x^2 + y^2 = z^2$
6. $px(x + y) = qy(x + y) - (x - y)(2x + 2y + z)$
7. $(y + zx)p - (x + yz)q = x^2 - y^2$
8. $p + 3q = 5z + \tan(y - 3x)$
9. $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$
10. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$
11. $x(z - 3y^3)p + y(3x^3 - z)q = 3(y^3 - x^3)z$
12. $(y^2 - x)p + yq = z + xy$
13. $xu_x + yu_y + zu_z = xyz$
14. $xu_x + 2yu_y + 3zu_z + 4tu_t = 0$
15. Find the integral surface of the partial differential equation
$$(3 - 2yz)p + x(2z - 1)q = 2x(y - 3)$$
which passes through the circle $z = 0$, $x^2 + y^2 = 4$.
16. Find the integral surface of the partial differential equation
$$2y(z - 3)p + (2x - z)q = y(2x - 3)$$
which passes through the circle $z = 0$, $x^2 + y^2 = 2x$.
17. Find the integral surface of the partial differential equation
$$(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$$
which passes through the curve $xz = a^3$, $y = 0$.
18. Find the integral surface of the partial differential equation
$$(2y^2 + z)p + (y + 2x)q = 4xy - z$$
which passes through the straight line $z = 1$, $y = x$.
19. Find the integral surface of the partial differential equation
$$y(x - z)p + (z^2 - xz - x^2)q = y(2x - z)$$
which passes through the ellipse $z = 0$, $2x^2 + 4y^2 = 1$.
20. Find the integral surface of the partial differential equation
$$(x - y)p + (y - x - z)q = z$$
which passes through the circle $z = 1$, $x^2 + y^2 = 1$.

19.5 NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER. CHARPIT'S METHOD

The general non-linear partial differential equation of the first order is of the type

$$F(x, y, z, p, q) = 0, \qquad ...(19.73)$$

where the function F is not necessarily linear in p and q. Before introducing a general method of solving Eq. (19.73), we will discuss the different forms of the solutions of this equation. The various types of solutions are

1. Complete integral, or Complete solution: We have seen in Section (19.2) that elimination of the arbitrary constants a and b from the equation

$$f(x, y, z, a, b) = 0 \qquad ...(19.74)$$

results in a partial differential equation of the form (19.73). It can be shown that the converse also holds, that is, any partial differential equation of the form (19.73) has solution of the form (19.74).

Any two parameters system of surfaces of the form (19.74) which satisfies the partial differential equation (19.73) is called a 'complete integral', or 'complete solution' of this equation.

2. General solution, or General integral: If we obtain a one parameter family of surfaces

$$f(x, y, z, a, \psi(a)) = 0, \qquad ...(19.75)$$

a subsystem of surfaces (19.74) by choosing $b = \psi(a)$ and form its envelope by eliminating the arbitrary constant 'a' from (19.75) and the equation $\partial f/\partial a = 0$, then we obtain a solution of the partial differential equation (19.73). When the function $\psi(a)$ is arbitrary this solution obtained is called the *'general solution'* or *'general integral'* of the Eq., (19.73). When a definite function $\psi(a)$ is used then the solution obtained is called a *'particular integral'* of the Eq. (19.73).

3. Singular integral, or Singular solution: The envelope of the two-parameter systems of surfaces (19.74) obtained by eliminating a and b from the equations $f(x, y, z, a, b) = 0$, $\partial f/\partial a = 0$, $\partial f/\partial b = 0$, if it exists, is also a solution of Eq. (19.73). It is called the *'singular integral'*, or *'singular solution'* of the Eq. (19.73).

Next, we discuss a general method, called *Charpit's method*, to find a complete integral of the partial differential equation (19.73).

19.5.1 Charpit's Method

Consider the first order partial differential equation

$$f(x, y, z, p, q) = 0. \qquad ...(19.76)$$

The Charpit's method consists of finding another partial differential equation

$$g(x, y, z, p, q, a) = 0 \qquad ...(19.77)$$

which is compatible with the given Eq.(19.76), that is, the Jacobian

$$J = \frac{\partial(f, g)}{\partial(p, q)} = f_q g_p - g_q f_p \neq 0.$$

Thus the Eqs. (19.76) and (19.77) are solvable for p and q as

$$p = p(x, y, z, a) \text{ and } q = q(x, y, z, a),$$

where a is an arbitrary constant.

Since $z = z(x, y)$, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy. \qquad ...(19.78)$$

Substituting for $p = p(x, y, z, a)$ and $q = q(x, y, z, a)$ in (19.78) and then integrating the resultant equation, we obtain the solution of Eq. (19.76) consisting of two arbitrary constants a and b.

To determine the Eq. (19.77) we proceed as follows.

Using (19.87), it becomes

$$\frac{dx}{qxy} = \frac{dz}{2z^2} = \frac{dp}{2pz - pqy}. \qquad ...(19.89)$$

Each term in (19.89) is equal to $\frac{(dp/p - dz/z + dx/x)}{0}$, and thus

$$\frac{dp}{p} = \frac{dz}{z} - \frac{dx}{x}. \qquad ...(19.90)$$

Integrating (19.90), we obtain

$$p = \frac{az}{x}, \qquad ...(19.91)$$

where a is an arbitrary constant.

Substituting (19.91) in (19.87), we obtain

$$q = \frac{z}{ay}. \qquad ...(19.92)$$

Thus, $dz = p\,dx + p\,dy$ becomes $dz = \frac{az}{x}dx + \frac{z}{ay}dy$, which gives

$$\frac{dz}{z} = a\frac{dx}{x} + \frac{1}{a}\frac{dy}{y}. \qquad ...(19.93)$$

Integrating (19.93), we obtain

$$z = bx^a y^{1/a},$$

as the complete integral of Eq. (19.87), where a and b are arbitrary constants.

Example 19.16: Find a complete integral of the equation $z = p^2 + qy$. Also find the singular solution if it exists.

Solution: The equation is

$$p^2 + qy - z = 0. \qquad ...(19.94)$$

The Charpit's equations are

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2 + qy} = \frac{dp}{p} = \frac{dq}{0}.$$

From the last term of these equations, we obtain $q = a$.

Using it in (19.94), we obtain $p = \sqrt{z - qy} = \sqrt{z - ay}$.

Thus, $dz = p\,dx + q\,dy$ becomes

$$dz = \sqrt{z - ay}\,dx + a\,dy, \quad \text{or} \quad \frac{dz - a\,dy}{\sqrt{z - ay}} = dx.$$

Integrating it we obtain

$$2\sqrt{z - ay} = x + b, \quad \text{or} \quad z = ay + \frac{1}{4}(x + b)^2,$$

as the complete integral of (19.94), where a and b are arbitrary constants.

To find singular solution we obtain the envelope of the two parameters family

$$\phi(a, b) = z - ay - \frac{1}{4}(x + b)^2 = 0. \qquad ...(19.95)$$

Here, $\frac{\partial \phi}{\partial a} = 0$, gives $y = 0$, and

$$\frac{\partial \phi}{\partial b} = 0, \quad \text{gives } x = -b.$$

Substituting these in (19.95), we obtain $z = 0$ as a singular solution for the given partial differential equation.

Example 19.17: Find a complete integral of the equation $p^2x + q^2y = z$.

Solution: The equation is

$$p^2x + q^2y - z = 0. \qquad ...(19.96)$$

The Charpit's equations are

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2x + q^2y)} = \frac{dp}{p - p^2} = \frac{dq}{q - q^2}. \qquad ...(19.97)$$

From Eqs. (19.97), we have

$$\frac{p^2dx + 2px\,dp}{p^2x} = \frac{q^2dy + 2qy\,dy}{q^2y}, \text{ or } \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}, \text{ which gives}$$

$$p^2x = aq^2y, \qquad ...(19.98)$$

where a is an arbitrary constant.

Solving Eq. (19.96) and (19.98) for p, q, we have

$$p = \left[\frac{az}{(1 + a)x}\right]^{1/2} \text{ and } q = \left[\frac{z}{(1 + a)y}\right]^{1/2}.$$

Thus $dz = p\,dx + q\,dy$ becomes

$$dz = \left[\frac{az}{(1 + a)x}\right]^{1/2} dx + \left[\frac{z}{(1 + a)y}\right]^{1/2} dy$$

or, $$\left(\frac{1 + a}{z}\right)^{1/2} dz = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy.$$

Integrating it, we obtain

$$[(1 + a)z]^{1/2} = (ax)^{1/2} + y^{1/2} + b$$

as the complete integral of Eq. (19.96), where a and b are arbitrary constants.

Example 19.18: Find a singular solution, if it exists, of the equation

$$6yz - 6pxy - 3qy^2 + pq = 0. \qquad ...(19.99)$$

Solution: The Charpit's equations are

$$\frac{dx}{-(6xy-q)} = \frac{dy}{-(3y^2-p)} = \frac{dz}{-[p(6xy-q)+q(3y^2-p]} = \frac{dp}{-(-6py+6py)} = \frac{dq}{-(6z-6px-6qy+6qy)} \qquad ...(19.100)$$

The fourth term in (19.100) is $\frac{dp}{0}$, which implies $dp = 0$, that is $p = a$, where a is an arbitrary constant.

Using $p = a$ in Eq. (19.99) gives

$$q = \frac{6y(z-ax)}{3y^2-a}.$$

Thus $dz = p\,dx + q\,dy$ becomes

$$dz = a\,dx + \frac{6y(z-ax)}{3y^2-a}\,dy,$$

or,

$$\frac{d(z-ax)}{z-ax} = \frac{6y\,dy}{3y^2-a}.$$

Integrating it, we obtain

$$z - ax = b(3y^2 - a)$$

as the complete integral of (19.99), where a and b are arbitrary constants.

To find singular solution we are to obtain the envelope of the two parameter family

$$\phi(a, b) = z - ax - b(3y^2 - a) = 0. \qquad ...(19.101)$$

Here, $\frac{\partial\phi}{\partial a} = 0$, gives $-x + b = 0$, that is, $x = b$

and, $\frac{\partial\phi}{\partial b} = 0$, gives $-(3y^2 - a) = 0$, that is, $y^2 = a/3$.

Substituting these in (19.101), we obtain

$$z = ab = 3xy^2$$

as the singular solution of the given equation.

The singular solution is obtained by eliminating b from

$\phi(b) = (z + b^2)^2 - 4(b^2y + x) = 0$ and $\frac{\partial \phi}{\partial b} = (z + b^2) - 2y = 0$, which gives, $b^2 = (2y - z)$.

Substituting $b^2 = (2y - z)$ in (19.106) and simplifying, we obtain

$$y^2 - yz + x = 0.$$

as the particular integral of the given equation

EXERCISE 19.3

Find the complete integral of the partial differential equations

1. $z = p^2x + qy$
2. $p^2 + q^2 = x - y$
3. $(p^2 + q^2)y = qz$
4. $z = p^2x + q^2y$
5. $4x^2p^2 + 9y^2q^2 = z^2$
6. $2(z + xp + yq) = yp^2$
7. $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$
8. Verify that $(x - a)^2 + (y - b)^2 + z^2 = 1$ is the complete integral of the partial differential equation $z^2(p^2 + q^2 + 1) = 1$. Find its singular integral, if it exists.
9. Find the singular solution of the following differential equations, if exist

   (i) $px + qy + z = xq^2$ (ii) $z = px + qy + p^2 + q^2$
10. Find the particular integral of the differential equation $pq = z$ which passes through the parabola $x = 0$, $y^2 = z$.
11. Find the particular integral of the differential equation $px + q^2y = z$ which passes through the curve $x = 1$, $y + z = 0$.
12. Find the particular integral of the differential equation $z = p^2 - q^2$ which passes through the parabola $4z + x^2 = 0$, $y = 0$.

19.6 SOME SPECIAL FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

We consider a few special types of first-order partial differential equations which can be solved very easily by Charpit's method.

I. Equations containing p and q only, f(p, q) = 0: The equation is

$$f(p, q) = 0. \quad ...(19.107)$$

Charpit's equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}.$$

From the last two terms we can choose $p = a$, or $q = a$, the choice depends on the problem given. Say we choose $p = a$, then from (19.107)

$$f(a, q) = 0, \text{ or } \quad q = \phi(a), \quad \text{again a constant.}$$

Thus $dz = p\,dx + q\,dy$ becomes, $dz = a\,dx + \phi(a)dy$, which gives

$$z = ax + \phi(a)y + b,$$

as the complete integral of Eq. (19.107), where a and b are two arbitrary constants.

Example 19.20: Find a complete integral of the partial differential equation

$$\sqrt{p} + \sqrt{q} = 1.$$

Solutions: The equation involves only p and q so choose $p = a$. This gives

$$q = (1 - \sqrt{p})^2 = (1 - \sqrt{a})^2.$$

Hence the complete integral of the given equation is

$$z = ax + (1 - \sqrt{a})^2 y + b,$$

where a and b are two arbitrary constants.

II. Equations not containing the independent variable x and y; f(z, p, q) = 0: The equation is

$$f(z, p, q) = 0. \qquad \text{...(19.108)}$$

The Charpit's equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}.$$

The last two terms are, $dp/p = dq/q$, which give

$$p = aq, \qquad \text{...(19.109)}$$

where a is an arbitrary constant.

We solve (19.108) and (19.109) to obtain expressions for p and q, using these in $dz = p\,dx + q\,dy$ we find the complete integral.

Example 19.21: Find a complete integral of the partial differential equation

$$p^2 z + q^2 = 1. \qquad \text{...(19.110)}$$

Solution: The Eq. (19.110) is independent of variables x and y thus

$$p = aq, \qquad \text{...(19.111)}$$

where a is an arbitrary constant.

Solving (19.110) and (19.111) simultaneously for p and q, we obtain

$$p = a(1 + a^2 z)^{-1/2} \quad \text{and} \quad q = (1 + a^2 z)^{-1/2}.$$

Hence $dz = p\,dx + q\,dy$, becomes $(1 + a^2 z)^{1/2} dz = a\,dx + dy$.

Integrating, we obtain

$$\frac{2}{3}(1 + a^2 z)^{3/2} = a^3 x + a^2 y + a^2 b$$

as the complete integral, where a and b are two arbitrary constants.

Example 19.22: Find a complete integral of the equation $z^2 = 1 + p^2 + q^2$.

Solution: Equation is of the form $f(z, p, q) = 0$, which is independent of x and y and thus $p = aq$, where a is an arbitrary constant. Substituting this in the given equation we obtain

$$z^2 - 1 = q^2(1 + a^2),$$

which gives $q = \left[\dfrac{z^2-1}{1+a^2}\right]^{1/2}$, and hence $p = a\left[\dfrac{z^2-1}{1+a^2}\right]^{1/2}$.

Thus $dz = p\,dx + q\,dy$ becomes

$$(1 + a^2)^{1/2} \frac{dz}{(z^2-1)^{1/2}} = a\,dx + dy.$$

Integrating we obtain

$$(1 + a^2)^{1/2} \cosh^{-1} z = ax + y + b,$$

as a complete integral of the given equation, where a and b are arbitrary constants.

III. Separable form, $f(x, p) = g(y, q)$: The equation is

$$f(x, p) - g(y, q) = 0. \quad ...(19.112)$$

The Charpit's equations are

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{g_y}.$$

Consider the pair $dx/f_p = -dp/f_x$. It gives $f_x dx + f_p dp = 0$. Integrating we obtain

$$f(x, p) = a, \quad ...(19.113)$$

where a is an arbitrary constant.

Using (19.113) in (19.112) gives

$$g(y, q) = a \quad ...(19.114)$$

We solve for p and q from (19.113) and (19.114) respectively and use in $dz = p\,dx + q\,dy$ to find the complete integral.

Example 19.23: Find a complete integral of the partial differential equation $p^2 + q^2 = x + y$.

Solution: Equation is of separable form $p^2 - x = y - q^2$.

Hence p and q are given by $p^2 - x = a$ and $y - q^2 = a$, which give

$$p = (a + x)^{1/2} \text{ and } q = (y - a)^{1/2};$$

a being an arbitrary constant.

Thus, $dz = p\,dx + q\,dy$ becomes $dz = (a + x)^{1/2}dx + (y - a)^{1/2}dy$, which gives

$$z = (a + x)^{3/2} + (y - a)^{3/2} + b$$

as the complete integral of the given equation.

IV. Clairaut equation: It is of the form

$$z = px + qy + f(p, q) \quad ...(19.115)$$

The Charpit's equations are

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{px + qy + pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}.$$

From the last two terms we have $p = a$ and $q = b$, where a and b are two arbitrary constants.

Substituting these values in (19.115) we obtain

$$z = ax + by + f(a, b),$$

as the complete integral. The same can be verified by substitution.

Example 19.24: Find a complete integral of the equation

$$pqz = p^2(xq + 1) + q^2 (yp + 1).$$

Solution: Rewriting the given equation as

$$z = px + qy + \frac{p}{q} + \frac{q}{p},$$

which is of the Clairaut form. Hence the complete integral is

$$z = ax + by = \frac{a}{b} + \frac{b}{a},$$

where a and b are two arbitrary constants.

Certain partial differential equation can be reduced to one of the special types of partial differential equations just discussed, after some simple substitution. In the examples to follow we consider a few such equations.

Example 19.25: Find a complete integral of the equation $z^2(p^2x^2 + q^2) = 1$.

Solution: Equation is

$$z^2 \left[\left\{ \left(x\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1. \quad \text{...(19.116)}$$

Set $X = \ln x$, it gives $\frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial X} = x\frac{\partial z}{\partial x}$, thus Eq. (19.116) becomes

$$z^2 \left[\left(\frac{\partial z}{\partial X} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1, \quad \text{...(19.117)}$$

which is of the type $f(z, p, q) = 0$, taking $p = \partial z/\partial X$ and $z = z(X, y)$.

Hence $p = aq$, where a is an arbitrary constant.

Using this in (19.117) gives

$$q = \frac{1}{(\sqrt{1 + a^2})z} \quad \text{and} \quad p = \frac{a}{(\sqrt{1 + a^2})z}.$$

Thus $dz = p\,dX + q\,dy$, becomes $\sqrt{1 + a^2}\; z\,dz = a\,dX + dy$, which gives

$$\sqrt{(1 + a^2)}\, z^2 = 2(aX + y) + b,$$

or,

$$\sqrt{(1 + a^2)}\, z^2 = 2(a \ln x + y) + b$$

Hence, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}.\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}$, become respectively

$$p = z_u + yz_v \text{ and } q = z_u + xz_v.$$

Substituting for p and q in Eq. (19.119) and simplifying, we obtain

$$z_u = z_v^2 \qquad \text{...(19.120)}$$

which is of the form $f(p, q) = 0$.

Taking $z_v = a$ in (19.120) gives $z_u = a^2$, and hence the solution of (19.120) is

$$z = a^2u + av + b,$$

and, therefore, a complete integral of the given equation is

$$z = a^2(x + y) + axy + b,$$

where a and b are two arbitrary constants.

EXERCISE 19.4

Find complete integrals of the equations

1. $\sqrt{p} + \sqrt{q} = 1$
2. $p + q = pq$
3. $p^2 - 3q^2 = 5$
4. $zpq = p + q$
5. $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$
6. $2\sqrt{p} + 3\sqrt{q} = 6x + 2y$
7. $(p + q)(z - px - qy) = 1$
8. $p^3 + q^3 = 216z$
9. $z^2(p^2 + q^2) = x^2 + y^2$,
10. $2xyz = px^2y + qxy^2 + 4pq$
11. $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$

19.7 FINDING SURFACES ORTHOGONAL TO A GIVEN FAMILY OF SURFACES

An important application of the theory of linear partial differential equations of the first order is to find surfaces orthogonal to a given family of surfaces.

Let

$$f(x, y, z) = c \qquad \text{...(19.121)}$$

be the given one-parameter family of surfaces to which we find a system of surfaces which cut these surfaces at right angles and let the requisite surface be

$$z = g(x, y). \qquad \text{...(19.122)}$$

The direction ratios of the normals to (19.121) and (19.122) at the point of intersection $(x, y, z,)$ are respectively

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \text{ and } \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

EXERCISE 19.5

1. Find the general equation of surfaces orthogonal to the family given by $x(x^2 + y^2 + z^2) = cy^2$, where c is an arbitrary constant.
2. Find the surface which cuts orthogonally the family of surfaces $(2x + 3y)z = c(z + 2)$, where c is an arbitrary constant and which passes through the circle $3(x^2 + y^2) = 4,\ z = 1$.
3. Find the surface which is orthogonal to the family of surfaces $z = cxy(x^2 + y^2)$, where c is an arbitrary constant and which passes through the hyperbola $x^2 - y^2 = a^2,\ z = 0$.
4. Find the surface which is orthogonal to the family of surfaces $x^2 + y^2 + z^2 = cy$, $c \neq 0$ is arbitrary constant and which passes through the circle $x^2 + y^2 = 4,\ z = 1$.

19.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A *homogeneous linear partial differential equation of the nth order with constant coefficients is of the form*

$$(D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \dots + a_n D'^n)z = f(x, y), \qquad \text{...(19.130)}$$

where a_i's are constants. It is homogeneous in the sense that all terms contain derivatives of the same order.

Here, $D \equiv \dfrac{\partial}{\partial x}$ and $D' \equiv \dfrac{\partial}{\partial y}$.

The equation (19.130) can be written as

$$F(D, D')z = f(x, y). \qquad \text{...(19.131)}$$

As in case of ordinary differential equation, the complete solution of the Eq. (19.131) is the sum of the *'complementary function'*, a general solution of the equation

$$F(D, D')z = 0 \qquad \text{...(19.132)}$$

and the *'particular integral'*, a solution of the Eq. (19.131) not containing any arbitrary constant.

Further, *if z_i, $i = 1,2, \dots, n$ are n solutions of the Eq. (19.132), then $\sum_{r=1}^{n} c_r z_r$ where c_r's are arbitrary constants, is also a solution of Eq. (19.132).* The proof of this is on the lines as in case of ordinary differential equations.

We will assume the homogeneous differential operator $F(D, D')$ to be reducible, that is, it can be resolved into n linear factors of the form $(D - mD')$.

19.8.1 The Complementary Function

We explain the method by considering a homogeneous equation of order two given by

$$(D^2 + a_1 DD' + a_2 D'^2)z = 0. \qquad \text{...(19.133)}$$

however, the results are applicable to equations of higher orders also.

The auxiliary equation is

$$D^2 + a_1 DD' + a_2 D'^2 = 0. \qquad \text{...(19.134)}$$

Let its roots be $D/D' = m_1, m_2$.

Case I: When the roots are real and distinct

Then (19.133) can be written as

$$(D - m_1 D')(D - m_2 D')z = 0. \qquad \text{...(19.135)}$$

The Eq. (19.135) will be satisfied by the solution of the equation

$$(D - m_2 D')z = 0, \text{ or } \quad p - m_2 q = 0, \qquad \text{...(19.136)}$$

which is a linear partial differential equation of order one. The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$$

with two independent solutions as

$$y + m_2 x = k_1 \text{ and } \quad z = k_2,$$

where k_1, k_2 are two arbitrary constants.

Thus a solution of Eq. (19.135) is given by

$$k_2 = \phi(k_1), \text{ or } \quad z = \phi(y + m_2 x).$$

Similarly another solution of Eq. (19.135) corresponding to $(D - m_1 D')z = 0$ is given by

$$z = \psi(y + m_1 x).$$

Hence, the complete solution of Eq. (19.135) is

$$z = \psi(y + m_1 x) + \phi(y + m_2 x), \qquad \text{...(19.137)}$$

where ψ and ϕ are two arbitrary functions.

In case the auxiliary equation is of degree three with three distinct roots m_1, m_2 and m_3, then the complementary function will be

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x)$$

and so on.

Case II: When the roots are equal

In case the roots are equal, say $m_1 = m_2 = m$, then the Eq. (19.135) becomes

$$(D - mD')^2 z = 0. \qquad \text{...(19.138)}$$

Let $u = (D - mD')z$, then (19.138) can be written as $(D - mD')u = 0$, which as in Case I, gives

$$u = \phi(y + mx), \qquad \text{...(19.139)}$$

where ϕ is an arbitrary function, and thus

$$(D - mD')z = \phi(y + mx). \qquad \text{...(19.140)}$$

The Eq. (19.140) gives

$$p - mq = \phi(y + mx).$$

The corresponding subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y+mx)},$$

which give two independent solutions as

$y + mx = k_1$ and $z = \phi(k_1)x + k_2$ or, $z - x\phi(y + mx) = k_2$.

Thus the complete solution is $k_2 = \psi(k_1)$, that is,

$$z - x\phi(y + mx) = \psi(y + mx)$$

or, $$z = \psi(y + mx) + x\phi(y + mx), \qquad ...(19.141)$$

where ψ and ϕ are two arbitrary functions.

In case the auxiliary equation is of degree three and a root m is repeated thrice, then the solution corresponding to this is

$$z = \phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx),$$

and so on.

Example 19.29: Solve the equation $(D^2 + DD' - 2D'^2)z = 0$.

Solution: The auxiliary equation is

$$m^2 + m - 2 = 0$$

with roots $m = 1, -2$. Hence, the general solution is

$$y = \phi_1(y + x) + \phi_2(y - 2x),$$

where ϕ_1 , ϕ_2 are two arbitrary functions.

Example 19.30: Solve the equation $(D^3 - 6D^2D' + 11DD'^2 - 6D'^2)z = 0$.

Solution: The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0,$$

with roots $m = 1, 2, 3$. Hence, the general solution is

$$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x),$$

where ϕ_1, ϕ_2, ϕ_3 are three arbitrary functions.

Example 19.31: Solve the equation $(D^3 - 5D^2D' + 8DD'^2 - 4D'^3)z = 0$.

Solution: The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0,$$

with roots $m = 1,2, 2$. Hence, the general solution is

$$z = \phi_1(y + x) + \phi_2(y + 2x) + x\phi_3(y + 2x),$$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Example 19.32: Solve the biharmonic equation $(D^4 - 2D^2 D'^2 + D'^4)z = 0$.

Solution: The auxiliary equation is

$$m^4 - 2m^2 + 1 = 0, \text{ or } (m - 1)^2(m + 1)^2 = 0,$$

with roots $m = 1, 1, -1$ and -1. Hence the general solution is

$$z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - x) + x\,\phi_4(y - x),$$

where ϕ_i's are arbitrary functions.

19.8.2 The Particular Integral

Consider the equation $F(D,D')\,z = f(x, y)$, the particular integral is given as

$$z = \frac{1}{F(D, D')} f(x, y).$$

We find particular integrals for some specific forms of $f(x, y)$.

Case I: When $f(x, y) = e^{ax+by}$.

Since $De^{ax+by} = ae^{ax+by}$ and $D'e^{ax+by} = be^{ax+by}$, thus we can easily find that

$$F(D, D')e^{ax+by} = F(a, b)e^{ax+by}.$$

Hence we write

$$e^{ax+by} = [F(D, D')]^{-1}F(a, b)e^{ax+by}$$

or,
$$[F(D, D')]^{-1}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}, \qquad \text{...(19.142)}$$

provided $F(a, b) \neq 0$.

In case $F(a, b) = 0$, we may follow the general method to be discussed as **Case IV**.

Case II: When $f(x) = \sin(ax + by)$ or $\cos(ax + by)$

Since,
$$D^2 \sin(ax + by) = -a^2 \sin(ax + by)$$
$$DD' \sin(ax + by) = -ab \sin(ax + by)$$
$$D'^2 \sin(ax + by) = -b^2 \sin(ax + by)$$

and so on. Thus, we can easily find that

$$F(D^2, DD', D'^2) \sin(ax + by) = F(-a^2, -ab, -b^2) \sin(ax + by),$$

Hence,

$$\sin(ax + by) = [F(D^2, DD', D'^2)]^{-1}F(-a^2, -ab, -b^2) \sin(ax + by)$$

or,
$$[F(D^2, DD', D'^2)]^{-1} \sin(ax + b) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by), \qquad \text{...(19.143)}$$

provided $F(-a^2, -ab, -b^2) \neq 0$.

In case $F(-a^2, -ab, -b^2) = 0$, we may follow the general method to be discussed as **Case IV**.

Case III: When $f(x, y) = x^m y^n$, or a polynomial in x, y.

The particular integral is

$$z = [F(D, D')]^{-1}x^m y^n. \qquad \text{...(19.144)}$$

To evaluate it we expand $[F(D, D')]^{-1}$ as an infinite series in ascending powers of D or D', depending upon $m < n$ or $m > n$, and then operate on $x^m y^n$ term by term.

The particular integral is

$$z = \frac{1}{D^3 - 4D^2D' + 4DD'^2} \cos(2x + 3y)$$

$$= \frac{1}{(-4)D - 4(-6)D + 4(-9)D} \cos(2x + 3y)$$

$$= -\frac{1}{16D} \cos(2x + 3y) = -\frac{1}{32} \sin(2x + 3y).$$

Hence, the general solution is

$$z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - \frac{1}{32} \sin(2x + 3y).$$

Example 19.36: Solve $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$.

Solution: The auxiliary equation is

$$m^2 - 5m + 4 = 0, \text{ or } (m - 1)(m - 4) = 0$$

with roots $m = 1, 4$.

The complementary function is

$$z = \phi_1(y + x) + \phi_2(y + 4x),$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{D^2 - 5DD' + 4D'^2} \sin(4x + y).$$

Here $F(D, D') = 0$, for $D^2 = -16$, $DD' = -4$, $D'^2 = -1$. Thus the usual method fails. We adopt the general method.

The equation is

$$(D - D')(D - 4D')z = \sin(4x + y) \qquad \text{...(19.147)}$$

Set $u = (D - 4D')z$, the Eq. (19.147) becomes $(D - D')u = \sin(4x + y)$, which gives

$$u = \frac{1}{D - D'} \sin(4x + y)$$

$$= \int \sin(3x + c)\,dx, \text{ replacing } y \text{ by } c - x, \text{ since } m = 1$$

$$= -\frac{1}{3} \cos(3x + c) = -\frac{1}{3} \cos(4x + y).$$

Thus $(D - 4D)z = u$ gives

$$z = \frac{1}{D - 4D'}\left[-\frac{1}{3}\cos(4x + y)\right]$$

19.9 NON-HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

By a non-homogeneous linear partial differential equation we mean an equation of the form

$$F(D, D')z = f(x, y), \qquad \text{...(19.148)}$$

where the polynomial expression on the right side of (19.148) is not homogeneous one. As in case of homogeneous linear partial differential equations, the complete solution is the sum of the complementary function and the particular integer.

To find complementary function we shall assume that $F(D, D')$ is reducible into linear factors of the form $(D - mD' - a)$.

To solve $(D - mD' - a)z = 0$, we write it as

$$p - mq = az.$$

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}.$$

The integrals are $y + mx = c_1$ and $z = c_2e^{ax}$.

Writing the solution as $c_2 = \phi(c_1)$, we get

$$z = e^{ax}\,\phi(y + mx) \qquad \text{...(19.149)}$$

as a solution .

In case the factor is repeated twice, then we can show that the integral corresponding to, (by substituting $(D - mD' - a)\,z = u$), $(D - mD' - a)^2z = 0$ is

$$z = e^{ax}\phi_1(y + mx) + xe^{ax}\phi_2(y + mx), \qquad \text{...(19.150)}$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

In case the linear factor is $(D - a)$, then the solution (19.149) simplifies to

$$z = e^{ax}\phi(y)$$

and if the linear factor is $(D - mD')$, then (19.149) becomes

$$z = \phi(y + mx)$$

as seen in case of homogeneous equations.

But if the linear factor is of the form $(D' - b)$, then to find the solution consider the differential equation

$$(D' - b)z = 0, \text{ or } \quad q - bz = 0.$$

The auxiliary equations are

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{bz}$$

with solutions $x = k_1$, $z = k_2e^{by}$, and hence, the solution of $(D' - b)z = 0$ is

$$z = e^{by}\phi(x). \qquad \text{...(19.151)}$$

The complementary function of $F(D, D')z = f(x, y)$ is obtained as the sum of the corresponding solution to each linear factor as obtained above.

The method of obtaining particular integrals of non-homogeneous equations are similar to as in case of homogeneous one discussed in the preceding section and will be best explained in the examples to follow.

Example 19.39: Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$.

Solution: The auxiliary equation is

$$(D^2 - DD' + D' - 1) = 0, \text{ or } (D - 1)(D - D' + 1) = 0.$$

Hence complementary function is

$$z = e^x\phi_1(y) + e^{-x}\phi_2(y + x),$$

where ϕ_1, ϕ_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{D^2 - DD' + D' - 1}\cos(x + 2y)$$

$$= \frac{1}{-1 + 2 + D' - 1}\cos(x + 2y), \quad (D^2 = -1,\ DD' = -2)$$

$$= \frac{1}{D'}\cos(x + 2y) = \int \cos(x + 2y)dy = \frac{1}{2}\sin(x + 2y).$$

Hence the complete solution is

$$z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + \frac{1}{2}\sin(x + 2y).$$

Example 19.40: Solve $(D - 3D' - 2)^3 z = 6e^{2x}\sin(3x + y)$.

Solution: The auxiliary equation is

$$(D - 3D' - 2)^3 = 0.$$

Hence the complementary function is

$$z = e^{2x}\phi_1(y + 3x) + xe^{2x}\phi_2(y + 3x) + x^2e^{2x}\phi_3(y + 3x),$$

where ϕ_1, ϕ_2 and ϕ_3 are three arbitrary functions.

The particular integral is

$$z = \frac{1}{(D - 3D' - 2)^3}6e^{2x}\sin(3x + y) = 6e^{2x}\frac{1}{[(D + 2) - 3D' - 2]^3}\sin(3x + y)$$

$$= 6e^{2x}\frac{1}{(D - 3D')^3}\sin(3x + y) = 6e^{2x}\int\left(\int\left(\int \sin(3x + y)dx\right)dx\right)dx,$$

where y is to be replaced by $y - 3x$, since $m = 4$.

Thus, $$z = 6e^{2x}\int\left(\int\left(\int \sin c\, dx\right)dx\right)dx$$

$$= e^{2x}x^3 \sin c = x^3e^{2x} \sin (y + 3x),$$

taking the arbitrary constants zeros since we are finding particular integral.

Hence, the complete solution is

$$z = e^{2x}\phi_1(y + 3x) + xe^{2x}\phi_2(y + 3x) + x^2e^{2x}\phi_3(y + 3x) + x^3e^{2x} \sin (y + 3x).$$

Example 19.41: Solve $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$.

Solution: The auxiliary equation is

$$(D + D' - 1)(D + 2D' - 3) = 0.$$

Hence the complementary function is

$$z = e^x\phi_1(y - x) + e^{3x}\phi_2(y - 2x),$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{(D + D' - 1)(D + 2D' - 3)}(4 + 3x + 6y). \qquad ...(19.152)$$

Consider the operator

$$\begin{aligned}
\frac{1}{(D + D' - 1)(D + 2D' - 3)} &= \frac{1}{3}[1 - (D + D')]^{-1}\left[1 - \frac{D + 2D'}{3}\right]^{-1} \\
&= \frac{1}{3}[1 + (D + D') + \text{terms of higher orders}] \\
&\quad \times \left[1 + \frac{D + 2D'}{3} + \text{terms of higher orders}\right] \\
&= \frac{1}{3}\left[1 + (D + D') + \frac{D + 2D'}{3} + \text{terms of higher orders}\right] \\
&= \frac{1}{3}\left[1 + \frac{4D + 5D'}{3} + \text{terms of higher orders}\right].
\end{aligned}$$

Thus, (19.152) becomes

$$\begin{aligned}
z &= \frac{1}{3}\left[(4 + 3x + 6y) + \frac{1}{3}(4D + 5D')(4 + 3x + 6y)\right] \\
&= \frac{1}{3}\left[4 + 3x + 6y + \frac{1}{3}(12 + 30)\right] = x + 2y + 6.
\end{aligned}$$

Hence the complete solution is

$$z = e^x\phi_1(y - x) + e^{3x}\phi_2(y - 2x) + x + 2y + 6.$$

Example 19.42: Solve $(D + D' + 3)^2(D' - 2)z = e^{2x+y}$.

Solution: The auxiliary equation is

$$(D + D' + 3)^2(D' - 2) = 0.$$

Hence the complementary function is

$$z = e^{-3x}\phi_1(y - x) + xe^{-3x}\phi_2(y - x) + e^{2y}\phi_3(x),$$

where ϕ_1, ϕ_2, and ϕ_3 are arbitrary functions.

The particular integral is

$$z = \frac{1}{(D + D' + 3)^2 (D' - 2)} e^{2x+y} = \frac{1}{(2 + 1 + 3)^2 (-1)} e^{2x+y} = -\frac{1}{36} e^{2x+y}.$$

Hence the complete solution is

$$z = e^{-3x}\phi_1(y - x) + xe^{-3x}\phi_2(y - x) + e^{2y}\phi_3(x) - \frac{1}{36} e^{2x+y}.$$

Example 19.43: Solve $(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^4$.

Solution: The equation can be reduced to partial differential equation with constant coefficients by substitution

$$x = e^u \text{ and } y = e^v, \text{ so that } u = \ln x \text{ and } v = \ln y.$$

The dependent variable $z(x, y)$ becomes function of u and v and for simplicity we again denote it by $z(u, v)$. It is easy to verify that

$$yD'z = D_v z, \quad x^2D^2z = (D_u - 1) D_u z$$
$$xyDD'z = D_uD_vz \quad \text{and} \quad y^2D'^2z = (D_v - 1)D_v z,$$

where $D_u = \dfrac{\partial}{\partial u}$ and $D_v = \dfrac{\partial}{\partial v}$.

Thus the given equation becomes

$$[(D_u - 1)D_u - 4D_uD_v + 4(D_v - 1)D_v + 6D_v]z = e^{3u+4v}$$

or,

$$(D_u^2 - 4D_uD_v + 4D_v^2 - D_u + 2D_v)z = e^{3u+4v}$$

or,

$$(D_u - 2D_v)(D_u - 2D_v - 1)z = e^{3u+4v}.$$

The auxiliary equation is

$$(D_u - 2D_v)(D_u - 2D_v - 1) = 0$$

Hence the complementary function is

$$\begin{aligned} z &= \phi_1(v + 2u) + e^u\phi_2(v + 2u) \\ &= \phi_1(\ln y + 2 \ln x) + x\phi_2(\ln y + 2 \ln x) \\ &= f_1(yx^2) + x f_2(yx^2), \end{aligned}$$

where f_1, f_2 are two arbitrary functions.

The particular integral is

$$\begin{aligned} z &= \frac{1}{(D_u - 2D_v)(D_u - 2D_v - 1)} e^{3u+4v} \\ &= \frac{1}{(3 - 8)(3 - 8 - 1)} e^{3u+4v} = \frac{1}{30} e^{3u+4v} = \frac{1}{30} x^3y^4. \end{aligned}$$

where r, s, t, appear in the first degree and the coefficients R, S, T, V are functions of x, y, z, p and q only.

The method consists of reducing the Eq. (19.155) into an equivalent system of two equations from which we determine p, or q, or both p and q. If both p and q are determined, then we use $dz = pdx + qdy$ to find the solution, otherwise solution is obtained on the lines of solving a Lagrange's equation.

We have,

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy,$$

and

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy$$

which give respectively

$$r = \frac{dp - sdy}{dx} \quad \text{and} \quad t = \frac{dq - sdx}{dy}.$$

Substituting for r and t in (19.155) and rearranging the terms we obtain

$$(Rdpdy + Tdqdx - Vdxdy) + s(Sdxdy - R(dy)^2 - T(dx)^2) = 0. \qquad ...(19.156)$$

Since the Eq. (19.156) holds for arbitrary value of r, thus to hold this we must have

$$Rdpdy + Tdqdx - Vdxdy = 0 \qquad ...(19.157)$$

and,
$$R(dy)^2 - Sdydx + T(dx)^2 = 0. \qquad ...(19.158)$$

These equations are called *Monge's subsidiary equations.*

We can solve Eq. (19.158) as quadratic in dy/dx. We may get two distinct values for dy/dx. Both of these values may be used along with Eq. (19.157) to get two expressions involving p and q. These are solved for p and q and then substituting for p and q in $dz = pdx + qdy$, we solve for the general integral. Alternatively, we may use one of dy/dx along with Eq. (19.157) to get a first order linear partial differential equation, which is solved by Lagrange's method leading to the general solution.

In case both the values of dy/dx are equal then evidently we are left with the alternate approach only as discussed above.

Example 19.46: Solve $r + (a + b)s + abt = xy$ using Monge's method.

Solution: Here we have

$$R = 1, \quad S = (a + b), \quad T = ab, \quad V = xy.$$

The Monge's subsidiary equations

$$R(dy)^2 - Sdx\,dy + T(dx)^2 = 0, \text{ and } R\,dp\,dy + Tdqdx - Vdx\,dy = 0,$$

become respectively,

$$(dy)^2 - (a + b)dx\,dy + ab(dx)^2 = 0 \qquad ...(19.159)$$

and,
$$dp\,dy + (ab)dqdx - xy\,dx\,dy = 0. \qquad ...(19.160)$$

Factorizing (19.159), we get $(dy - adx)(dy - bdx) = 0$, which give

$$dy - adx = 0 \text{ and } dy - bdx = 0.$$

Integrating, we get respectively

$$y - ax = \text{const. and } y - bx = \text{const.}$$

Substituting $dy = adx$ in (19.160), we obtain

$$dp(adx) + ab\,dq\,dx - xy\,dx(adx) = 0$$

or, $$dp + bdq - xy\,dx = 0.$$

Integrating, we obtain

$$p + bq - y\frac{x^2}{2} = \text{const.} = \phi_1(y - ax), \qquad ...(19.161)$$

where ϕ_1 is an arbitrary function.

Similarly using $dy = bdx$ in (19.160) and integrating, we obtain

$$p + aq - y\frac{x^2}{2} = \text{const.} = \phi_2(y - bx), \qquad ...(19.162)$$

where ϕ_2 is an arbitrary function.

Solving (19.161) and (19.162) for p and q, we get

$$p = \frac{b\phi_2 - a\phi_1}{b - a} + \frac{x^2 y}{2} \text{ and } q = \frac{\phi_2 - \phi_1}{a - b}.$$

Substituting for p and q in $dz = pdx + qdy$, we have

$$dz = \left(\frac{b\phi_2 - a\phi_1}{b - a} + \frac{x^2 y}{2}\right)dx + \left(\frac{\phi_2 - \phi_1}{a - b}\right)dy$$

or, $$(a - b)dz = -\phi_1(dy - adx) + \phi_2(dy - bdx) + (a - b)\frac{x^2 y}{2}dx.$$

Integrating, we get

$$(a - b)z = -\int \phi_1(y - ax)d(y - ax) + \int \phi_2(y - bx)d(y - bx) + (a - b)\frac{y}{2}\int x^2 dx$$

or, $$z = f_1(y - ax) + f_2(y - bx) + \frac{1}{6}x^3 y,$$

where f_1 and f_2 are two arbitrary functions.

Remark. We can verify that the same solution is obtained by solving the equation $r + (a + b)s + abt = xy$ using the method of solving homogeneous equations as discussed in Section 19.8.

Example 19.47: Solve $x^2 r + 2xys + y^2 t = 0$ using Monge's method.

Solution: Here we have

$$R = x^2,\ S = 2xy,\ T = y^2,\ V = 0.$$

The Monge' subsidiary equations

$$R(dy)^2 - S\,dy\,dx + T(dx)^2 = 0 \quad \text{and} \quad R\,dp\,dy + T\,dq\,dx - V\,dy\,dx = 0,$$

become respectively

$$x^2(dy)^2 - 2xy\,dy\,dx + y^2(dx)^2 = 0 \qquad \text{...(19.163)}$$

and,

$$x^2dp\,dy + y^2dq\,dx = 0. \qquad \text{...(19.164)}$$

From Eq. (19.163), we have

$(xdy - ydx)^2 = 0$, or $xdy - ydx = 0$ with solution $y = ax$.

Substituting $y = ax$ in (19.164) to obtain $ax^2dp\,dx + a^2x^2dq\,dx = 0$, or $dp + adq = 0$, which gives

$$p + aq = \text{const.}$$

or, $p + (y/x)\,q = \phi_1(y/x)$, or $xp + yq = x\phi_1(y/x)$

which is Lagrange's equation.

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x\phi_1(y/x)}.$$

From the first two terms, we get $y = k_1\,x$ and from the first and the last term, we have

$$z = x\phi_1(k_1) + k_2 = x\phi_1(y/x) + k_2.$$

Hence the solution can be obtained as

$$z = x\phi_1(y/x) + \phi_2(y/x),$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

Remark. We note that the equation $x^2r + 2xys + y^2t = 0$ can be reduced to *l.p.d.e.* with constant coefficients by using the transformation $x = e^u$ and $y = e^v$ and the same solution can be obtained by solving the equation using the method as discussed in Section 19.8.

Example 19.48: Solve $(x - y)(xr - xs - ys + yt) = (x + y)(p - q)$

Solution: Here $R = x(x - y)$, $S = -(x^2 - y^2)$, $T = y(x - y)$, $V = (x + y)(p - q)$.

The Monge's equations

$$R(dy)^2 - S\,dy\,dx + T(dx)^2 = 0 \quad \text{and} \quad R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0,$$

become respectively

$$x(x - y)(dy)^2 + (x^2 - y^2)dy\,dx + y(x - y)(dx)^2 = 0$$

and,

$$x(x - y)dp\,dy + y(x - y)dq\,dx - (x + y)(p - q)dx\,dy = 0$$

Simplifying these we get respectively

$$x(dy)^2 + (x + y)dy\,dx + y(dx)^2 = 0 \qquad \text{...(19.165)}$$

and,

$$x\,dp\,dy + y\,dq\,dx - \frac{x+y}{x-y}(p - q)dx\,dy = 0 \qquad \text{...(19.166)}$$

The Eq. (19.165) may be factorized as $(x\,dy + y\,dx)(dx + dy) = 0$, which gives

$$xy = \text{const. or,} \quad x + y = \text{const.}$$

Taking xy = constant and dividing each term of (19.166) by $x\,dy$, or its equivalent $-y dx$, we get

$$dp - dq - \frac{p-q}{x-y}(dx - dy) = 0$$

or,
$$\frac{dp - dq}{p - q} = \frac{dx - dy}{x - y}$$

$$\frac{(p-q)}{(x-y)} = \text{const.} = \phi_1(xy), \quad \text{(say)}$$

or,
$$p - q = (x - y)\phi_1(xy),$$

a Lagrange's linear equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x-y)\phi_1(xy)}.$$

The first two terms give $x + y = c$, c being a constant.

The first and the third terms give

$$\begin{aligned} dz &= (x - y)\,\phi_1(xy)dx \\ &= -(c - 2x)\phi_1(xc - x^2)dx \\ &= -\phi_1(cx - x^2)d(cx - x^2). \end{aligned}$$

Integrating, we have

$$z = f_1(cx - x^2) + c'.$$

Hence the solution can be given as

$$\begin{aligned} z &= f_1(cx - x^2) + f_2(c), \\ &= f_1(xy) + f_2(x + y), \end{aligned}$$

where f_1 and f_2 are two arbitrary functions.

Example 19.49: Solve the equation $q^2 r - 2pqs + p^2 t = 0.$

Solution: Here, $R = q^2$, $S = -2pq$, $T = p^2$ and $V = 0$.

The Monge's equations

$$R(dy)^2 - S dx\,dy + T(dx)^2 = 0 \text{ and } R dp\,dy + T dq\,dx - V dx\,dy = 0,$$

become respectively,

$$q^2(dy)^2 + 2pq\,dx\,dy + p^2(dx)^2 = 0 \qquad \text{...(19.167)}$$

and,
$$q^2 dp\,dy + p^2 dq\,dx = 0. \qquad \text{...(19.168)}$$

From (19.167), we have

$$(pdx + qdy)^2 = 0, \text{ or } \quad dz = 0, \text{ which gives, } z = k_1.$$

Dividing Eq. (19.168) by qdy, or its equivalent $-pdx$, we obtain

$$qdp = pdq, \text{ which gives } \quad p = k_2 q, \text{ or } p = \phi_1(k_1)q$$

Thus $p = \phi_1(z)q$, where ϕ_1 is an arbitrary function.

The equation $p - q\,\phi_1(z) = 0$ is Lagrange's equation with subsidiary equations

$$\frac{dx}{1} = \frac{dy}{-\phi_1(z)} = \frac{dz}{0},$$

with integrals $z = k_3$ and $y + x\phi_1(k_3) = k_4 = \phi_2(k_3)$, say

Thus the general solution is

$$y + x\phi_1(z) = \phi_2(z),$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

Example 19.50: Solve the equation $r - t\cos^2 x + p\tan x = 0$.

Solution: Here, $R = 1$, $S = 0$, $T = -\cos^2 x$ and $V = -p\tan x$

The Monge's equations

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0 \text{ and } R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0,$$

become respectively

$$(dy)^2 - \cos^2 x(dx)^2 = 0, \qquad \ldots(19.169)$$

and

$$dp\,dy - \cos^2 x\,dq\,dx + p\tan x\,dx\,dy = 0. \qquad \ldots(19.170)$$

From (19.169), we have $dy = \cos x\,dx$ and $dy = -\cos x\,dx$

The equation $dy = \cos x\,dx$ gives $y - \sin x = k_1$.

Using $dy = \cos x\,dx$ in (19.170), and simplifying we obtain

$$dp - \cos x\,dq + p\tan x\,dx = 0$$

or,
$$\sec x\,dp - dq + p\tan x\sec x\,dx = 0$$

or,
$$d(p\sec x) - dq = 0.$$

Integrating, $p\sec x - q = k_2 = \phi(k_1)$, where ϕ is arbitrary, this gives

$$p\sec x - q = \phi\,(y - \sin x)$$

a Lagrange's equation.

The subsidiary equations are

$$\frac{dx}{\sec x} = \frac{dy}{-1} = \frac{dz}{\phi(y - \sin x)}.$$

The first two terms give, $y + \sin x = k_3$. Also from the last two terms, we obtain

$$dz = -\phi(y - \sin x)dy = -\phi(2y - k_3)dy = -\frac{1}{2}\phi(2y - k_3)d(2y),$$

which gives
$$z = -\frac{1}{2}\int \phi(2y - k_3)d(2y) + k_4$$
$$= \phi_1(2y - k_3) + \phi_2(k_3), \quad \text{say},$$

9. $z^2 = x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\ln\dfrac{x+\sqrt{x^2+a}}{y+\sqrt{y^2-a}} + 2b$

10. $z = \sqrt{a(x+y)} + \sqrt{(1-a)(x-y)} + b$. 11. $z = ax^2 + by^2 + 8ab$

Exercise 19.5 (p. 262)

1. $x^2 + y^2 + z^2 = zf\left(\dfrac{2x^2+y^2}{z^2}\right)$
2. $3(x^2+y^2) = z^2(z+3)$
3. $(x^2+y^2+4z^2)(x^2-y^2)^2 = a^4(x^2+y^2)$
4. $x^2+y^2+z^2 = 5z$.

Exercise 19.6 (p. 270)

1. $z = \phi_1(y) + \phi_2(y+x) + \phi_3(y+2x)$
2. $z = \phi_1(y-2x) + \phi_2(2y-x)$
3. $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$
4. $z = \phi_1(x+y) + x\phi_2(y+x) + \phi_3(x-2y) + x\phi_4(x-2y)$
5. $z = \phi_1(y-2x) + \phi_2(y-3x) + e^{x-y}/2$
6. $z = \phi_1(2y-3x) + x\phi_2(2y-3x) + x^2e^{3x-2y}/8$
7. $z = \phi_1(y-3x) + \phi_2(2y+x) + \sin(2x-y)/5$
8. $z = \phi_1(y+x) + \phi_2(y-x) + [2x\sin(x+y) + \cos(x+y)]/4$
9. $z = \phi_1(y-x) + \phi_2(y+2x) + ye^x$
10. $z = \phi_1(y+x) + x\phi_2(y+x) + \dfrac{x^2}{2}\tan(y+x)$
11. $z = \phi_1(y-2x) + \phi_2(y+x) - \left[4x^2 + \dfrac{1}{22}(x+5y)^2\{2\ln(x+5y)-1\}\right]$
12. $z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y$
13. $z = \phi_1(y+5x) + \phi_2(y-3x) + x^4 + 2x^3y$
14. $z = \phi_1(y+x) + \phi_2(y-x) + (\tan x\tan y)/2$
15. $z = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x) + e^x(\cos 2y + 2\sin 2y)/25$.

Exercise 19.7 (p. 276)

1. $z = \phi_1(x) + \phi_2(y) + e^{3x}\phi_3(y+2x)$
2. $z = \phi_1(y+x) + e^{-x}\phi_2(y-x)$
3. $z = \sum c_k e^{a_kx+b_ky}$, provided $a_k^2 - 4b_k^2 + 2a_k + 1 = 0$
4. $z = e^{-x}\phi_1(y-x) + xe^{-x}\phi_2(y-x)$
5. $z = \phi_1(y+2x) + e^{-x/2}\phi_2(y) + xe^{-x/2}\phi_3(y)$

6. $z = e^{-x}\phi_1(y - x) + xe^{-x}\phi_2(y - x) + e^{2x+y}/16$
7. $z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + [2\cos(x + 2y) - 3\sin(x + 2y)]/39$
8. $z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - ye^{x+2y} - \dfrac{1}{3}\left(\dfrac{x^2y}{2} + \dfrac{xy}{3} + \dfrac{x^2}{2} + \dfrac{2x}{9}\right)$
9. $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (\sin(x + 2y))/2 - xe^y$
10. $z = e^{2x}[x^2\tan(y + 2x) + x\phi_1(y + 3x) + \phi_2(y + 3x)]$
11. $z = \dfrac{1}{82}[\sin(x - 3y) + 9(x - 3y)] + \sum c_k e^{a_kx+b_ky};\ a_k - b_k^2 = 2$
12. $z = \dfrac{A}{m^2 - l^2}\{m\sin(lx + my) + l^2\cos(lx + my)\} + \sum_k c_k e^{a_kx+b_ky};\quad a_k^2 - b_k = 0$
13. $z = \sum c_k e^{a_kx+b_ky} - \left(y^2 + \dfrac{x^4}{12}\right);\qquad a_k^2 - b_k = 0$
14. $z = \phi_1\left(\dfrac{x}{y}\right) + \phi_2(xy) + \dfrac{1}{6}(\ln x)^2$
15. $z = \phi_1(xy) + \phi_2(x^2y) + x + y.$

Exercise 19.8 (p. 282)

1. $z = \phi_1(y + ax) + \phi_2(y - ax)$
2. $z = \phi_1(y + \ln x) + x\phi_2(y + \ln x)$
3. $z = y^3 + y\phi_1(y^2 + 2x) + \phi_2(y^2 + 2x)$
4. $z = \phi_1(x^2y) + \phi_2(xy^2)$
5. $y = xz + \phi_1(z) + \phi_2(x)$
6. $y = \phi_1(z) + z\phi_2(x)$
7. $y = \phi_1(x + z) + \phi_2(z)$
8. $z = \phi_1(x + \tan y) + \phi_2(x - \tan y)$
9. $x = \phi_1(y + 2x - z) + y\phi_2(y + 2x - z)$
10. $y = \phi_1(x + y + z) + x\phi_2(x + y + z).$

20 Applications of Partial Differential Equations

CHAPTER

"The scope of applications of partial differential equations is much wider as compared to ordinary differential equations. They arise in many diversified areas like epidemiology, traffic flow studies and economic analysis. Models of three major kinds of physical phenomena: wave motion, heat conduction and potential theory lead respectively to three important partial differential equations, the wave equation, the heat equation and the Laplace's equation. Various mathematical tools including Fourier series, integrals, transforms and special functions are employed to solve these equations".

20.1 METHOD OF SEPARATION OF VARIABLES

To find the solution of a particular problem involving a partial differential equation, depending upon the nature of the physical phenomena, it is necessary to specify that the solution satisfies some specific conditions. In case these conditions are imposed on spatial boundaries belonging to the region D, where the solution is required, then such conditions are called *boundary conditions* and the problem is called *boundary value problem* (BVP). However, in case one of the independent variable is time, then it becomes necessary to specify how the solution starts and a condition of this type is called an *initial condition* and the problem is called *initial value problem* (IVP). Problems, with both initial and boundary conditions specified, are called *initial boundary value problems* (IBVP).

Method of separating variables, or '*product method*' is a powerful technique to solve linear partial differential equations with specified conditions. For a *PDE* in the unknown function u of two independent variables x and y, we assume that the desired solution is separable, that is,

$$u(x, y) = X(x)Y(y),$$

where X is a function of x alone and Y that of y alone. The substitution of u and its partial derivatives in terms of X and Y and their derivatives reduce the given *PDE* to the form

$$f(X, X', X'', ...) = g(Y, Y', Y'', ...)$$

which is separable in X and Y. Here primes denote the derivatives w.r.t. the corresponding independent variables.

Since f is a function of x alone and g is a function of y alone, x and y being independent, thus it follows that each must be equal to a common constant and hence the problem of solving the *PDE* reduces to finding the solutions to two *ODE's* given by

$$f(X, X', X'' \ldots) = 0, \quad g(Y, Y', Y'' \ldots) = 0$$

under specified conditions. From the expressions of $X(x)$ and $Y(y)$ so obtained we find expression for $u(x, y)$, the desired solution.

Example 20.1: Solve by the method of separation of variables the initial value problem (IVP)

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0; \quad u(0, y) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(0, y) = e^{-3y}, \text{ for all } y.$$

Solution: Let the solution be

$$u(x, y) = X(x)Y(y). \qquad \ldots(20.1)$$

It gives $\dfrac{\partial u}{\partial x} = X'Y, \quad \dfrac{\partial u}{\partial y} = XY'$ and $\dfrac{\partial^2 u}{\partial x^2} = X''Y,$

where the primes denote the derivatives w.r.t. the corresponding independent variable.

Substituting these in the given equation, we obtain

$$X''Y - 2X'Y + XY' = 0$$

or,
$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y}. \qquad \ldots(20.2)$$

Since x and y are independent variables, thus Eq. (20.2) can hold only if each side is equal to some constant, say k. Hence we get

$$X'' - 2X' - kX = 0 \qquad \ldots(20.3)$$

and,
$$Y' + kY = 0 \qquad \ldots(20.4)$$

two ordinary differential equations.

The solution of Eq. (20.3) is

$$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$$

and of Eq. (20.4) is

$$Y = c_3\, e^{-ky},$$

where c_1, c_2 and c_3 are arbitrary constants.

Substituting for X and Y in (20.1), the solution of the given equation is

$$u(x, y) = \left[Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}\right]e^{-ky}, \qquad \ldots(20.5)$$

where $A = c_1c_3$ and $B = c_2c_3$.

The three arbitrary constants A, B and k can be determined from the conditions specified.

Using $u(0, y) = 0$ in (20.5) gives, $(A + B)\, e^{-ky} = 0$ for all y, which implies

$$A + B = 0.$$

Similarly using $\frac{\partial u}{\partial x}(0, y) = e^{-3y}$ in (20.5) gives, $[(1 + \sqrt{1+k})A + (1 - \sqrt{1+k})B]e^{-ky} = e^{-3y}$, for all y.

This implies $(1 + \sqrt{1+k})A + (1 - \sqrt{1+k})B = 1$ and $k = 3$, and hence

$$3A - B = 1.$$

Solving for A and B, we obtain $A = -B = \frac{1}{4}$. Thus (20.5) becomes

$$u(x, y) = \frac{1}{4}(e^{3x} - e^{-x})e^{-3y},$$

the desired solution of the given *IVP*.

EXERCISE 20.1

Solve the following equations by the method of separation of variables:

1. $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$

2. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$

3. $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u;\quad u(x, 0) = 6e^{-3x}$

4. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u;\quad u(0, y) = 0,\quad \frac{\partial u(0, y)}{\partial x} = 1 + e^{-3y}$

5. $\frac{\partial^2 u}{\partial x \partial y} = e^{-y}\cos x;\quad u(x, 0) = 0$ and $\frac{\partial u}{\partial y}(0, y) = 0.$

20.2 VIBRATING STRING: ONE-DIMENSIONAL WAVE EQUATION

The transversal oscillations induced in a guitar or violin string are governed by a partial differential equation called the *one-dimensional wave equation*.

Consider an elastic string placed along the x-axis from 0 to l, fixed at the ends $x = 0$ and $x = l$. We distort it at some instant $t = 0$ and then release to allow it to vibrate in the xy-plane. Let the function $u(x, t)$ denotes the displacement of the string at any point x and at any instant $t > 0$, as shown in Fig. 20.1.

To determine the mathematical model, the *p.d.e* in the displacement function $u(x, t)$, of the physical system described we make the following assumptions:

(i) The mass of the string per unit length ρ is constant and the string is perfectly elastic.

(ii) The tension caused by stretching the string is so large that the gravitational force on the string can be neglected, and

(iii) The string performs small transverse motion in a vertical plane e.g., xy-plane in Fig. (20.1) and thus the displacement $u(x, t)$ and the slope at every point of the string always remains small in magnitude.

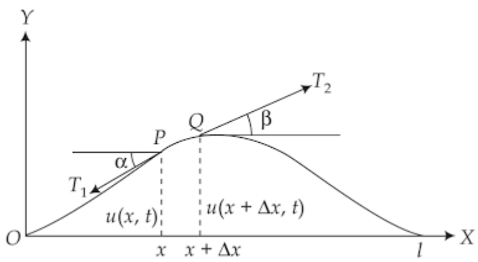


Fig. 20.1

Consider a segment PQ of the string between x and $x + \Delta x$ and let T_1 and T_2 be the tensions at the end-points P and Q of the segment acting along the tangents to the curve of the string at that points.

Since the motion of the string is only in the vertical plane, thus the horizontal components of the tension must be constant. Hence with the notations as in Fig. 20.1, we have

$$T_1 \cos \alpha = T_2 \cos \beta = \text{const. say, } T. \qquad \text{...(20.6)}$$

In the vertical direction we have two forces $-T_1 \sin \alpha$ at P and $T_2 \sin \beta$ at Q with upward direction taken as positive. By Newton's second law of motion, the resultant of the two forces must be equal to mass $\rho \Delta x$ of the segment PQ multiplied with the acceleration $\dfrac{\partial^2 u}{\partial t^2}$ evaluated at some point between the segment PQ, that is

$$T_2 \sin \beta - T_1 \sin \alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}. \qquad \text{...(20.7)}$$

Dividing (20.7) by (20.6), we obtain

$$\tan \beta - \tan \alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}. \qquad \text{...(20.8)}$$

Since $\tan \alpha$ and $\tan \beta$ are the slopes to the curve of the string at $P(x)$ and $Q(x + \Delta x)$, thus

$$\tan \alpha = \left.\left(\frac{\partial u}{\partial x}\right)\right|_{x} \quad \text{and} \quad \tan \beta = \left.\left(\frac{\partial u}{\partial x}\right)\right|_{x+\Delta x}.$$

Using these in (20.8), we obtain

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}.$$

Taking the limit as $\Delta x \to 0$, we obtain

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{...(20.9)}$$

where $c^2 = \rho / T$, since $\rho / T > 0$.

The Eq. (20.9) is called the *one-dimensional wave equation*. It is called one-dimensional, since it involves only one space variable. The equation is second order homogeneous and of hyperbolic type, refer Section (19.1) .

In order to solve the wave equation (20.9) we must incorporate the conditions imposed by the physical model under study.

If the ends of the string are fixed, then

$$u(0, t) = u(l, t) = 0, \text{ for } t \geq 0.$$

These are the *boundary conditions*.

The *initial conditions* specifies the initial position

To solve (20.15) and (20.16), we consider the following three cases.

(i) When k is positive, say $k = \lambda^2$, then

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}; \quad T = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t},$$

(ii) When k is negative, say $k = -\lambda^2$, then

$$X = c_5 \cos \lambda x + c_6 \sin \lambda x; \quad T = c_7 \cos c\lambda t + c_8 \sin c\lambda t,$$

(iii) When k is zero, then

$$X = c_9 x + c_{10}; \quad T = c_{11} t + c_{12}.$$

Thus, the various possible solutions of the wave equation are:

$$u = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \qquad \text{...(20.17)}$$

$$u = (c_5 \cos \lambda x + c_6 \sin \lambda x)(c_7 \cos c\lambda t + c_8 \sin c\lambda t) \qquad \text{...(20.18)}$$

$$u = (c_9 x + c_{10})(c_{11} t + c_{12}). \qquad \text{...(20.19)}$$

In case of vibrating string problem u is a periodic funtion of x and t. Hence the solution (20.18) is the proper one, since it involves trigonometric expressions in x and t. We write

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos c\lambda t + D \sin c\lambda t) \qquad \text{...(20.20)}$$

as the suitable solution of the wave equation (20.10), where A, B, C and D are arbitrary constants to be determined using the initial and boundary conditions.

Using the boundary condition $u(0, t) = 0$ in (20.20), gives

$$A[C \cos c\lambda t + D \sin c\lambda t] = 0, \text{ for all } t,$$

which implies that $A = 0$.

Hence, (20.20) becomes

$$u(x, t) = B \sin \lambda x[C \cos c\lambda t + D \sin c\lambda t]. \qquad \text{...(20.21)}$$

Using the second boundary condition $u(l, t) = 0$ in (20.21), gives

$$B \sin \lambda l[C \cos c\lambda t + D \sin c\lambda t] = 0, \text{ for all } t$$

Now B can't be zero, since the solution will become zero one, thus $\sin \lambda l = 0 = \sin n\pi$, and hence, $\lambda = \frac{n\pi}{l}$, n being an integer.

Therefore, the solution (20.21) can be expressed as

$$u_n(x, t) = \left[C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l}\right] \sin\left(\frac{n\pi x}{l}\right).$$

Using the principle of superposition, we obtain

$$u(x, t) = \sum_{n=1}^{\infty} \left[C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l}\right] \sin\left(\frac{n\pi x}{l}\right), \qquad \text{...(20.21a)}$$

where the constants are to be evaluated using the initial conditions (20.12).

Using the initial condition $u(x, 0) = f(x)$ in (20.21a), we obtain

Example 20.3: A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $v_0x(l - x)$, find the displacement of the string at any distance x from one end at any time t.

Solution: The displacement function $u(x, t)$ is given by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \qquad ...(20.34)$$

with boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0 \qquad ...(20.35)$$

and initial conditions

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_0x(l - x), \quad 0 \leq x \leq l. \qquad ...(20.36)$$

The solution of the wave equation (20.34) subject to the boundary conditions (20.35) and the initial condition $u(x, 0) = 0$, refer Eq. (20.28) is given by

$$u(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right), \qquad ...(20.37)$$

where the constants D_n, $n = 1, 2, ...$ are to be determined using the second initial condition $\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_0x(l - x)$.

Differentiating (20.37) w.r.t. t and substituting $t = 0$ and using $\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_0x(l - x)$, it becomes

$$u_0x(l - x) = \sum_{n=1}^{\infty} D_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l}. \qquad ...(20.38)$$

Next, expanding $u_0x(l - x)$ in a half-range sine series in $[0, l]$, we have

$$u_0x(l - x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad ...(20.39)$$

where $b_n = \frac{2}{l}\int_0^l u_0x(l - x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2u_0}{l}\left[-\frac{l}{n\pi}x(l - x)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l - 2x)\sin\frac{n\pi x}{l} - 2\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4u_0 l^2}{n^3\pi^3}[1 - \cos n\pi] = \frac{4u_0 l^2}{\pi^3 n^3}[1 - (-1)^n], \text{ and hence}$$

$$b_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8u_0 l^2}{\pi^3 n^3}, & \text{when } n \text{ is odd.} \end{cases} \qquad \text{...(20.40)}$$

From (20.38), (20.39) and (20.40), we have

$$D_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8u_0 l^3}{\pi^4 n^4 c}, & \text{when } n \text{ is odd} \end{cases}$$

Substituting for D_n in (20.37) the required solution is given by

$$u(x, t) = \sum_{n=1,3,5}^{\infty} \frac{8u_0 l^3}{\pi^4 n^4 c} \sin\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{8u_0 l^3}{\pi^4 c} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi ct}{l} \sin\frac{(2n-1)\pi x}{l}.$$

Example 20.4: The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.

Solution: Let P and Q be the points of trisection of the string OA of length say l, fixed at the points O and A as shown in Fig. 20.2. Initially, it is taken to the form $OP'Q'A$, where $PP' = QQ' = a$, say and is then released.

The displacement $u(x, t)$ of string at a distance x from the fixed point O at $t > 0$ is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(20.41)}$$

with boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0 \qquad \text{...(20.42)}$$

and initial conditions

$$u(x, 0) = \begin{cases} \dfrac{3a}{l}x, & 0 \leq x \leq \dfrac{l}{3} \\ \dfrac{3a}{l}(l - 2x), & \dfrac{l}{3} \leq x \leq \dfrac{2l}{3} \\ \dfrac{3a}{l}(x - l), & \dfrac{2l}{3} \leq x \leq l \end{cases} \qquad \text{...(20.43)}$$

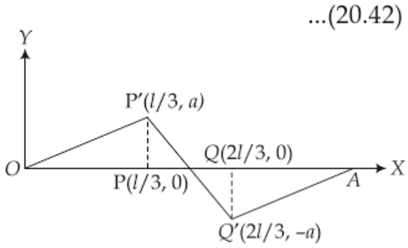


Fig. 20.2

and, $$\left.\frac{\partial u}{\partial x}\right|_{t=0} = 0. \qquad \text{...(20.43a)}$$

The solution of the one-dimensional wave equation (20.41) subject to the boundary conditions (20.42) and the initial condition (20.43a), refer Eq. (20.27) is given by

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}, \qquad \text{...(20.44)}$$

where C_n are constants to be determined using the initial condition at (20.43).

Substituting $t = 0$ in (20.44), we obtain

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}, \quad 0 \le x \le l. \qquad \text{...(20.45)}$$

The series (20.45) is a Fourier half-range sine series expansion of $u(x, 0)$ in $(0, l)$, and hence the coefficients C_n's are given by

$$C_n = \frac{2}{l} \int_0^l u(x, 0) \sin \frac{n\pi x}{l}\, dx.$$

In order for solution to satisfy the initial condition (20.43), we must have

$$C_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l} dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l-2x) \sin \frac{n\pi x}{l} dx + \int_{2l/3}^{l} \frac{3a}{l}(x-l) \sin \frac{n\pi x}{l} dx\right]$$

$$= \frac{6a}{l^2}\left[\left[x\left(\frac{-\cos(n\pi x/l)}{n\pi/l}\right) - 1.\left(\frac{-\sin(n\pi x/l)}{(n\pi/l)^2}\right)\right]_0^{l/3} + \left[(l-2x)\left(-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right) - (-2)\left(-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right)\right]_{l/3}^{2l/3}\right.$$

$$\left. + \left[(x-l)\left(-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right) - (1)\left(-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right)\right]_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left(\sin \frac{n\pi}{3} - \sin \frac{2n\pi}{3}\right), \text{ after simplification.}$$

$$= \frac{18a}{n^2\pi^2} \sin \frac{n\pi}{3}[1 + (-1)^n], \text{ since } \sin \frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n \sin \frac{n\pi}{3}.$$

This gives

$$C_n = \begin{cases} 0, & \text{when } n \text{ is odd.} \\ \dfrac{36a}{n^2\pi^2} \sin \dfrac{n\pi}{3}, & \text{when } n \text{ is even.} \end{cases}$$

Substituting the value for C_n in (20.44), we obtain the desired solution as

$$u(x, t) = \sum_{n=2,4,6}^{\infty} \frac{36a}{n^2\pi^2} \sin \frac{n\pi}{3} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l},$$

or,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{9a}{n^2\pi^2} \sin \frac{2n\pi}{3} \cos \frac{2n\pi ct}{l} \sin \frac{2n\pi x}{l}. \qquad \text{...(20.46)}$$

The displacement of the mid-point is obtained from (20.46) by substituting $x = l/2$, which gives $u(l/2, t) = 0$, since $\sin \dfrac{2n\pi x}{l} = 0$ at $x = l/2$ for all n. Hence the mid-point of the string is always at rest.

20.4 *D'* ALEMBERT'S SOLUTION OF THE WAVE EQUATION

The *D'* Alembert's method consists of finding the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(20.47)}$$

directly by change of variables.

Consider the transformation

$$\xi = x + ct \text{ and } \eta = x - ct. \qquad \text{...(20.48)}$$

It gives

$$\xi_x = \eta_x = 1 \text{ and } \xi_t = -\eta_t = c.$$

Also $u(x, t)$ becomes a function of ξ and η. By chain rule, we have

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta, \\ u_{xx} &= (u_\xi + u_\eta)_\xi \xi_x + (u_\xi + u_\eta)_\eta \eta_x \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}. \end{aligned} \qquad \text{...(20.49)}$$

Similarly,

$$u_t = c(u_\xi - u_\eta)$$

and,

$$u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}). \qquad \text{...(20.50)}$$

Substituting from (20.49) and (20.50) in Eq. (20.47), we obtain

$$u_{\xi\eta} = \frac{\partial^2 u}{\partial \xi \partial \eta} = 0, \qquad \text{...(20.51)}$$

which can be solved directly by two successive integrations.

Integrating (20.51) with respect to ξ, we obtain

$$\frac{\partial u}{\partial \eta} = h(\eta),$$

where $h(\eta)$ is an arbitrary function of η. Integrating next with respect to η, we obtain

$$u = \int h(\eta)d\eta + \phi(\xi), \qquad ...(20.52)$$

where $\phi(\xi)$ is an arbitrary function of ξ.

Taking $\psi(\eta) = \int h(\eta)d\eta$, (20.52) becomes

$$u = \phi(\xi) + \psi(\eta),$$

or,
$$u = \phi(x + ct) + \psi(x - ct), \qquad ...(20.53)$$

using (20.48).

The solution (20.53) is known as '*D' Alembert's solution of the one-dimensional wave equation*.

In case the initial conditions are

$$u(x, 0) = f(x), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x),$$

then differentiating (20.53) w.r.t. t we obtain

$$\frac{\partial u}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$$

and hence we have

$$\phi(x) + \psi(x) = f(x) \quad \text{and} \quad c\phi'(x) - c\psi'(x) = g(x)$$

or,
$$\phi'(x) + \psi'(x) = f'(x) \quad \text{and} \quad \phi'(x) - \psi'(x) = \frac{1}{c}g(x).$$

Solving these for $\phi'(x)$ and $\psi'(x)$, we obtain

$$\phi'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x),$$

and,
$$\psi'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x).$$

Integrating, we obtain

$$\left.\begin{aligned} \phi(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds + k(x_0) \\ \psi(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds - k(x_0) \end{aligned}\right\} \qquad ...(20.54)$$

where x_0 is arbitrary and k is a constant chosen such that $\phi(x) + \psi(x) = f(x)$.

Substituting (20.54) in (20.53), we obtain

Example 20.5: Use *D*'Alembert's method to find the displacement of a vibrating string with initial velocity zero and initial displacement $f(x) = k(\sin x - \sin 2x)$.

Solution: If $u(x, t)$ is the displacement, then

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with initial conditions

$$u(x, 0) = f(x) = k(\sin x - \sin 2x) \text{ and } \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = 0.$$

By *D*' Alembert's method, refer (20.26), the solution is

$$\begin{aligned} u(x, t) &= \frac{1}{2}[f(x + ct) + f(x - ct)] \\ &= \frac{1}{2}[k\{\sin(x + ct) - \sin 2(x + ct) + \sin(x - ct) - \sin 2(x - ct)\}] \\ &= \frac{k}{2}[\{\sin(x + ct) + \sin(x - ct)\} - \{\sin 2(x + ct) + \sin 2(x - ct)\}] \\ &= k[\sin x \cos ct - \sin 2x \cos 2ct]. \end{aligned}$$

Obviously the given conditions are satisfied by the solution obtained.

Example 20.6: Use *D*' Alembert's method to find displacement of a vibrating string with initial displacement $f(x) = \sin x$ and initial velocity $g(x) = a$, where a is a constant.

Solution: If $u(x, t)$ is the displacement, then

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with initial conditions

$$u(x, 0) = f(x) = \sin x \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = a.$$

By *D*' Alembert's method, refer (20.55), the solution is

$$\begin{aligned} u(x, t) &= \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)dx \\ &= \frac{1}{2}[\sin(x + ct) + \sin(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} a\,dx \\ &= \sin x \cos ct + at. \end{aligned}$$

EXERCISE 20.2

1. In the vibrating string problem an elastic string of length l is fixed at $x = 0$ and at $x = l$. It is taken to the position $f(x) = A \sin \dfrac{2\pi x}{l}$ at $t = 0$ and then released. Find the displacement function of the string motion.
2. An elastic string of length l is fixed at both ends at $x = 0$ and $x = l$. At a distance 'a' units from the end $x = 0$, the string is transversely displaced to a distance 'd' and is released from rest when it is in that position. Find the expression for the displacement function $u(x, t)$.
3. An elastic string of length l which is fastened at its ends $x = 0$ and $x = l$ is released from its equilibrium position with initial velocity $g(x)$ given as

$$g(x) = \begin{cases} x, & 0 \le x \le l/3 \\ 0, & l/3 < x \le l. \end{cases}$$

Find the displacement of the string at any instant of time t.

4. Solve, $\quad \dfrac{\partial^2 u}{\partial t^2} = 9\dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0$

given that $\quad y(0,t) = y(2, t) = 0, \quad t \ge 0$

and, $\quad y(x, 0) = x(x - 2), \quad \left.\dfrac{\partial y}{\partial t}\right|_{t=0} = g(x), \quad 0 \le x \le 2,$

where $\quad g(x) = \begin{cases} 0, & 0 \le x < 1/2, \quad 1 < x \le 2 \\ 3, & 1/2 \le x \le 1 \end{cases}.$

5. Solve $\quad \dfrac{\partial^2 u}{\partial t^2} = 8\dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < 4, \quad t > 0$

given that, $\quad y(0,t) = y(4,t) = 0, \quad t \ge 0$

and $\quad y(x, 0) = x^2(x - 4), \quad \left.\dfrac{\partial y}{\partial t}\right|_{t=0} = 1, \quad 0 \le x \le 4.$

6. An elastic string of length 20 cm fixed at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity given by

$$g(x) = \begin{cases} x, & 0 \le x \le 10 \\ 20 - x, & 10 < x \le 20 \end{cases},$$

x being the distance from one end. Determine the displacement function at any time t.

7. Show that the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

subject to the conditions

$$u(0, t) = u(l, t) = 0, \quad t \geq 0, \quad u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

has the solution

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n t + B_n \sin \lambda_n t) \sin \frac{n\pi x}{l}\ ; \lambda_n = \frac{n\pi x}{l},$$

where $\quad A_n = \dfrac{2}{l}\displaystyle\int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad$ and $\quad B_n = \dfrac{2}{cn\pi}\displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l} dx$.

8. Use D' Alembert's method to find the displacement function of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection

(i) $f(x) = a(x - x^3)$ (ii) $f(x) = a \sin^2 \pi x$.

9. Use D' Alembert method to find the solution of the initial value problem defining the vibrations of an infinitely long elastic string when

(i) $f(x) = 0, \quad g(x) = \sin 3x$,

(ii) $f(x) = \sin 2x, \quad g(x) = \cos 2x$,

(iii) $f(x) = e^{-|x|}, \quad g(x) = \cos 4x$.

10. Solve the vibrating string problem when there is a resistance in the medium which is proportional to the velocity. If the initial displacement is $f(x)$ and the string starts from rest, then the *bvp* is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0, \quad 0 < a < 1$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0;$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi.$$

20.5 ONE-DIMENSIONAL HEAT FLOW EQUATION

Consider a straight thin bar or wire of constant cross-sectional area 'A' and of homogeneous material placed along the x-axis from 0 to l. Assume that the bar is insulated laterally and there is no heat flow along the surface. Thus, the heat flow is only along x-axis perpendicular to the cross-section of the bar. Let the density ρ(gm/cm^3), specific heat s(cal/gm. deg) and the thermal conductivity k(cal/cm deg.sec) are constants and let $u(x, t)$ be the temperature at a distance x from O at time t.

Consider a typical segment of the bar of thickness δx between x and $x + \delta x$, as shown in Fig. 20.3, and let δu be the temperature change in this segment.

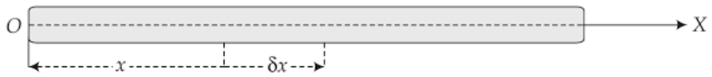


Fig. 20.3

If R_1 and R_2 respectively are the rates (cal/sec) of inflow and outflow of heat respectively at x and $x + \delta x$, then

$$R_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad R_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

and hence the net rate at which the heat enters this segment of the bar at time t is

$$R_1 - R_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right].$$

Under the assumption that there is no source or sink of heat in this segment, this must be equal to the rate at which the heat energy accumulates in this segment of width δx which is $s\rho A\delta x \dfrac{\partial u}{\partial t}$.

Hence, we have

$$\frac{\partial u}{\partial t} = \frac{k}{s\rho}\left[\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right].$$

Taking the limit as $\delta x \to 0$, we get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{...(20.59)}$$

where $c^2 = \dfrac{k}{s\rho}$ (cm^2/sec) is called the *diffusivity* of the bar depending on its material.

This is *one-dimensional heat equation*.

The heat equation (20.59) together with certain initial and boundary conditions, which are associated with the physical models, uniquely determines the temperature distribution throughout the bar at any time $t > 0$.

For example, we may have the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

with boundary conditions

$$u(0, t) = T_1, \quad u(l, t) = T_2, \quad t \geq 0,$$

and, initial conditions

$$u(x, 0) = f(x), \quad 0 \leq x \leq l.$$

20.6 SOLUTION OF THE HEAT EQUATION BY SEPARATION OF VARIABLES AND USE OF FOURIER SERIES

We solve the heat equation (20.59) for some specific boundary and initial conditions, using separation of variables and Fourier series. We consider the following cases.

Case I: *Ends of the bar kept at temperature zero*

If $u(x, t)$ is the temperature distribution in a thin, homogeneous bar of length l with both ends kept at zero temperature and $f(x)$ is the initial temperature distribution in the bar, then the boundary value problem modelling $u(x, t)$ is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \qquad \text{...(20.60)}$$

$$u(0, t) = u(l, t) = 0, \; t \geq 0 \qquad \text{...(20.61)}$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq l. \qquad \text{...(20.62)}$$

Substituting $u(x, t) = X(x)T(t)$ in Eq. (20.60), we get

$$XT' = c^2 X''T$$

or,
$$\frac{T'}{c^2 T} = \frac{X''}{X}, \qquad \text{...(20.63)}$$

where the primes denote the derivatives w.r.t. the corresponding independent variables.

The left side of Eq. (20.63) depends only on t and the right side only on x, and since x and t are independent, so the both sides must be equal to a constant k. Thus (20.63) leads to two ordinary differential equations

$$X'' - kX = 0, \qquad \text{...(20.64)}$$

and,
$$T' - kc^2 T = 0. \qquad \text{...(20.65)}$$

To solve (20.64) and (20.65), we consider the following three cases.

(i) When k is positive, say $k = p^2$, then

$$X = c_1 e^{px} + c_2 e^{-px}; \quad T = c_3 e^{c^2 p^2 t}.$$

(ii) When k is negative, say $k = -p^2$, then

$$X = c_4 \cos px + c_5 \sin px, \quad T = c_6 e^{-c^2 p^2 t}.$$

(iii) When k is zero, then

$$X = c_7 x + c_8, \quad T = c_9.$$

Thus the various possible solutions of the heat equation are:

$$u(x, t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{c^2 p^2 t}) \qquad \text{...(20.66)}$$

$$u(x, t) = (c_4 \cos px + c_5 \sin px)(c_6 e^{-c^2 p^2 t}) \qquad \text{...(20.67)}$$

$$u(x, t) = (c_7 x + c_8) c_9. \qquad \text{...(20.68)}$$

Out of these we have to choose the solution consistent with the physical constraints of the model under study. Since we are dealing with the problem on heat condition the solution must be transient one, that is, u decreases with time t; and hence the solution (20.67) is the suitable one. We write

$$\frac{\partial u\,(0, t)}{\partial x} = \frac{\partial u\,(l, t)}{\partial x} = 0, \quad t > 0 \qquad ...(20.73)$$

and initial condition

$$u(x, 0) = f(x), \quad 0 \le x \le l. \qquad ...(20.74)$$

The appropriate solution of the heat equation (20.72), refer (20.69), is

$$u(x, t) = (A \cos px + B \sin px)e^{-c^2p^2t}. \qquad ...(20.75)$$

Differentiating (20.75) w.r.t. x, we obtain

$$\frac{\partial u}{\partial x} = (-A \sin px + B \cos px)pe^{-c^2p^2t}. \qquad ...(20.76)$$

Using the condition $\frac{\partial u\,(0, t)}{\partial x} = 0$ in (20.76), we obtain $Bpe^{-c^2p^2t} = 0$, for all t. Since $p \neq 0$, thus $B = 0$, hence Eq. (20.76) becomes

$$\frac{\partial u}{\partial x} = -Ap \sin px\, e^{-c^2p^2t}. \qquad ...(20.77)$$

Next using $\frac{\partial u\,(l, t)}{\partial x} = 0$ in (20.77), we obtain

$$-Ap \sin pl\, e^{-c^2p^2t} = 0, \text{ for } t > 0.$$

Since for non-zero solution of $u(x, t)$, A can't be zero and also $p \neq 0$ hence $\sin pl = 0$, which gives, $p = \frac{n\pi}{l}$, n being an integer. Hence the solution of the heat equation subject to the boundary conditions (20.73) is

$$u_n(x, t) = A_n \cos \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l^2}}, \quad n \text{ being an integer.}$$

By the superposition principle the solution is

$$u(x, t) = {\sum_{n=0}^{\infty}}' A_n \cos \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l^2}}, \quad 0 < x < l, \quad t > 0. \qquad ...(20.78)$$

Applying the initial condition (20.74), we obtain

$$u(x, 0) = f(x) = {\sum_{n=0}^{\infty}}' A_n \cos \frac{n\pi x}{l}, \quad 0 < x < l,$$

which is the Fourier half-range cosine series of $f(x)$ in $(0, l)$. Hence the coefficients A_n are the Fourier coefficients given by

$$A_0 = \frac{1}{l}\int_0^l f(x)dx, \quad A_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l} dx, \text{ n} = 1, 2, \ldots \qquad ...(20.79)$$

Solution: The temperature distribution $u(x, t)$ is modelled as the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 100, \quad t > 0 \qquad ...(20.84)$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u(100, t) = 0, \quad t \geq 0 \qquad ...(20.85)$$

and the initial condition

$$f(x) = u(x, 0) = \begin{cases} x, & 0 \leq x \leq 50 \\ 100 - x, & 50 \leq x \leq 100. \end{cases} \qquad ...(20.86)$$

The solution of the heat equation (20.84) subject to the boundary conditions (20.85), refer (20.70), is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{100} e^{-\frac{c^2 n^2 \pi^2 t}{(100)^2}}, \quad 0 \leq x \leq 100, \quad t \geq 0. \qquad ...(20.87)$$

Using the initial condition (20.86), (20.87) gives

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{100}, \quad 0 \leq x \leq 100,$$

which is Fourier half-range sine series expansion of $f(x)$ in the interval [0, 100]. Hence using (20.86) the coefficients B_n are given by

$$B_n = \frac{2}{100} \int_0^{100} f(x) \sin \frac{n\pi x}{100} dx$$

$$= \frac{1}{50} \left[\int_0^{50} x \sin \frac{n\pi x}{100} dx + \int_{50}^{100} (100 - x) \sin \frac{n\pi x}{100} dx \right].$$

Integrating and simplifying, we obtain

$$B_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \frac{400}{n^2 \pi^2}, & \text{if } n \text{ is odd} \end{cases}.$$

Substituting for B_n in (20.87) we get

$$u(x, t) = \frac{400}{\pi^2} \left[\sin \frac{\pi x}{100} e^{-\frac{c^2 \pi^2}{(100)^2} t} - \frac{1}{9} \sin \frac{3\pi x}{100} e^{-\frac{9c^2 \pi^2}{(100)^2}} + \frac{1}{25} \sin \frac{5\pi x}{100} e^{\frac{-25 c^2 \pi^2 t}{(100)^2}} - \ldots \right],$$

as the desired solution.

Example 20.9: Find the temperature distribution in a laterally insulated bar of length l with both the ends insulated and initial temperature in the rod being $\sin(\pi x/l)$.

Solution: If $u(x, t)$ is the temperature distribution in the bar, then it is modelled by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(20.88)$$

subject to the boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0, \qquad ...(20.89)$$

and initial condition

$$u(x, 0) = \sin\left(\frac{\pi x}{l}\right). \qquad ...(20.90)$$

The solution of the heat equation (20.88) subject to the boundary conditions (20.89), refer Eq. (20.78), is

$$u(x, t) = {\sum_{n=0}^{\infty}}' A_n \cos\frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2}{l^2} t}. \qquad ...(20.91)$$

Using the initial condition (20.90), (20.91) becomes

$$\sin\left(\frac{\pi x}{l}\right) = {\sum_{n=0}^{\infty}}' A_n \cos\frac{n\pi x}{l},$$

which is Fourier half-range cosine series of $\sin\frac{\pi x}{l}$ in the interval $[0, l]$. Hence

$$A_0 = \frac{1}{l}\int_0^l \sin\frac{\pi x}{l} dx \quad \text{and} \quad A_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l} \cos\frac{n\pi x}{l} dx$$

We have, $A_0 = \frac{1}{l}\left[-\frac{l}{\pi}\cos\frac{\pi x}{l}\right]_0^l = -\frac{1}{\pi}\left[\cos\frac{\pi x}{l}\right]_0^l = \frac{2}{\pi}$,

$$A_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l} \cos\frac{\pi x}{l} dx = 0.$$

$$A_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l} \cos\frac{n\pi x}{l} dx = \frac{1}{l}\int_0^l \left[\sin\frac{(1+n)\pi x}{l} + \sin\frac{(1-n)\pi x}{l}\right]$$

$$= -\frac{1}{l}\left[\frac{l}{(1+n)\pi}\cos\frac{(1+n)\pi x}{l} + \frac{l}{(1-n)\pi}\cos\frac{(1-n)\pi x}{l}\right]_0^l, \quad n \neq 1$$

$$= -\frac{[(-1)^{n+1}-1]}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{2[(-1)^{n+1}-1]}{\pi(n^2-1)}, \quad n \neq 1$$

Thus, $$A_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{-4}{\pi(n^2-1)}, & \text{if } n \text{ is even.} \end{cases}$$

Hence (20.91) becomes

$$u(x, t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos\frac{2n\pi x}{l}\, e^{-\frac{4c^2n^2\pi^2}{l^2}t},$$

the required temperature distribution.

Example 20.10: (a) A laterally insulated bar of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to and maintained at 0°C, find the temperature distribution in the rod at a distance x from A at time t.

(b) In case the temperature at A is raised to 20°C and that at B is reduced to 80°C, then what will be the temperature distribution.

Solution: (a) If $u(x, t)$ is the temperature distribution at a distance x from the fixed end A at any time $t > 0$, then it is modelled as the one-dimensional heat equation given by

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0. \qquad \text{...(20.92)}$$

Now prior to the temperature change at the end B, the heat flow was independent of time, that is, the system was in steady state, and hence the Eq. (20.92) was reduced to $\frac{\partial^2 u}{\partial x^2} = 0$, which gives

$$u = ax + b, \qquad \text{...(20.93)}$$

where a and b are arbitrary constants. Using the conditions $u(0) = 0$ and $u(l) = 100$, we obtain from (20.93), $a = \frac{100}{l}$ and $b = 0$, and hence the initial condition is given by

$$f(x) = u(x, 0) = \frac{100}{l}x. \qquad \text{...(20.94)}$$

Also the boundary conditions for the subsequent flow are

$$u(0, t) = 0 \text{ and } u(l, t) = 0, \text{ for } t > 0. \qquad \text{...(20.95)}$$

Thus, the temperature distribution will be the solution of Eq. (20.92) subject to initial condition (20.94) and boundary conditions (20.95).

The solution of the one-dimensional heat equation (20.92) subject to the boundary conditions (20.95), refer Eq. (20.70), is

and, $$\left.\begin{aligned} u_t(0, x) &= u(0, t) - u_s(0) = 20 - 20 = 0 \\ u_t(l, x) &= u(l, t) - u_s(l) = 80 - 80 = 0. \end{aligned}\right\} \quad ...(20.101)$$

Also, $u_t(x, 0) = u(x, 0) - u_s(x) = \dfrac{100x}{l} - \left(\dfrac{60x}{l} + 20\right) = \dfrac{40x}{l} - 20.$ ...(20.102)

Now, we solve for the transient temperature function $u_t(x, t)$ as the solution of the heat equation (20.92) subject to the boundary conditions (20.101) and the initial condition (20.102).

As in part (a), the solution satisfying (20.101) is

$$u_t(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2}{l^2} t} \quad ...(20.103)$$

using the initial condition (20.102) in (20.103), we obtain

$$\left(\frac{40x}{l} - 20\right) = u_t(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}, \quad 0 < x < l$$

which is the half-range Fourier sine series expansion of the function $f(x) = 20\left(\dfrac{2x}{l} - 1\right)$ in the interval $(0, l)$ and hence the coefficients B_n are given by

$$B_n = \frac{2}{l}\int_0^l 20\left(\frac{2x}{l} - 1\right) \sin \frac{n\pi x}{l} dx$$

$$= \frac{40}{l}\left[\left(-\frac{l}{n\pi}\left(\frac{2x}{l} - 1\right)\cos \frac{n\pi x}{l}\right) + \left(\frac{l}{n\pi}\right)^2 . \frac{2}{l} \sin \frac{n\pi x}{l}\right]_0^l$$

$$= -\frac{40}{n\pi}[1 + \cos n\pi] = -\frac{40}{n\pi}[1 + (-1)^n].$$

Thus, $$B_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\dfrac{80}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

and hence (20.103) becomes

$$u_t(x, t) = -\frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l} e^{-\frac{4c^2 n^2 \pi^2 t}{l^2}}. \quad ...(20.104)$$

Thus the desired solution is obtained by combining (20.100) and (20.104) as

$$u(x, t) = u_s(x) + u_t(x, t)$$

$$= 20 + \frac{60}{l}x - \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l} e^{-\frac{4c^2 n^2 \pi^2 t}{l^2}}.$$

EXERCISE 20.3

1. Find the temperature distribution $u(x, t)$ in a laterally insulated bar 80 cm long if the initial temperature is $100 \sin\left(\dfrac{3\pi x}{80}\right)$°C and the ends are kept at 0°C. How long will it take for the maximum temperature in the ball to drop to 50°C if the density, specific heat and thermal conductivity for the bar material are 8.92 gm/cm^3, 0.092 cal/gm°C and 0.5 cal/cm sec °C, respectively?
2. A rod of length l laterally insulated is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature distribution in the rod at any time t.
3. Find the temperature distribution in a laterally insulated bar of length l whose ends are kept at temperature 0°C assuming that the initial temperature is

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le l/2 \\ l - x, & \text{if } l/2 \le x \le l \end{cases}$$

4. Find the temperature distribution $u(x, t)$ in a laterally insulated bar of length 10 cm whose ends are kept at temperature 0°C when the initial temperature is $f(x) = x(10 - x)$ in °C. It is given that density is 10.6 gm/cm^3, specific heat is 0.056 cal/gm deg and thermal conductivity is 1.04 cal/cm deg.
5. A bar of length l laterally insulated has its ends A and B kept at 0° and u_0° respectively until steady-state conditions prevail. If the temperature at B is then suddenly reduced to 0° and kept so while that of B is maintained at 0°, find the temperature distribution in the bar at any subsequent time.
6. A bar 100 cm long, laterally insulated, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution at any subsequent time. Also show that the sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.
7. Solve $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, such that

   (i) u is not infinite, when $t \to \infty$,

   (ii) $\partial u/\partial x = 0$, when $x = 0$ and $u(x, t) = 0$, when $x = l$, for all t,

   (iii) $u(x, 0) = u_0$, for all t.
8. A bar AB of length 10 cm laterally insulated has its ends A and B kept at 30° and 100° temperatures respectively until steady-state prevails. Then the temperature at A is lowered to 20° and that of B to 40° and the ends are maintained at these temperatures. Find the temperature distribution in the bar at the subsequent time.

A steady state two-dimensional heat flow problem consists of Laplace Eq. (20.108) to be considered in some region R of the xy-plane along with the boundary conditions, say $u(x, y) = f(x, y)$ on the boundary curve of R. This forms the boundary value problem called the *Dirichlet problem*. The difficulty of a Dirichlet problem depends on the form of region R. We will consider this for rectangular and circular regions only.

20.8 SOLUTION OF LAPLACE'S EQUATION IN CARTESIAN COORDINATES

The equation modelling the steady state temperature distribution $u(x, y)$ in the rectangular region is given by the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b. \qquad ...(20.111)$$

We solve this equation by applying the method of separation of variables.

Let the solution be of the form

$$u(x, y) = X(x)Y(y). \qquad ...(20.112)$$

Substituting (20.112) in Eq. (20.111), we get

$$X''Y + XY'' = 0$$

or,
$$\frac{X''}{X} = -\frac{Y''}{Y}, \qquad ...(20.113)$$

where primes denote the derivatives w.r.t. the corresponding independent variables.

Since X''/X is a function of x only and Y''/Y is a function of y only and, x, y being independent, thus (20.113) holds when each side of (20.113) is equal to a constant, say k. Hence we have two differential equations:

$$X'' - kX = 0 \text{ and } \quad Y'' + kY = 0.$$

To solve these equations, we consider the following three cases.

(i) When k is positive, say $k = p^2$, then

$$X = c_1e^{px} + c_2e^{-px}, \quad Y = c_3 \cos py + c_4 \sin py$$

(ii) When k is negative, say $k = -p^2$, then

$$X = c_5 \cos px + c_6 \sin px, \quad Y = c_7e^{py} + c_8e^{-py}$$

(iii) When k is zero, then

$$X = c_9\, x + c_{10}, \quad Y = c_{11}y + c_{12}.$$

Thus the three possible solutions of the Laplace equation (20.111) are

$$u(x, y) = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(20.114)$$

$$u(x, y) = (c_5 \cos px + c_6 \sin px)\,(c_7e^{py} + c_8e^{-py}) \qquad ...(20.115)$$

$$u(x, y) = (c_9x + c_{10})\,(c_{11}y + c_{12}). \qquad ...(20.116)$$

Of these we select the solution compatible with the boundary conditions given.

In the examples to follow we shall find the solution of the Laplace equation subject to different boundary conditions.

$$u(x, b) = u_0 \sin \frac{\pi x}{a}.$$

Using this in (21.120), we obtain

$$u_0 \sin \frac{\pi x}{a} = \sum_{n=1}^{\infty} 2B_n \sin \frac{\pi x}{a} \sinh \frac{n\pi b}{a}$$

Comparing coefficients on both sides we obtain

$$B_1 = \frac{u_0}{2 \sinh (\pi b/a)} \quad \text{and} \quad B_2 = B_3 = ... = 0$$

and hence (20.120) becomes

$$u(x, y) = \frac{u_0}{\sinh (\pi b/a)} \sin \frac{\pi x}{b} \sinh \frac{\pi y}{a},$$

the desired solution.

Example 20.12: Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \qquad ...(20.121)$$

subject to the boundary conditions

$$u(x, 0) = u(x, b) = 0 \quad \text{and} \quad u(0, y) = 0, u(a, y) = \pi y(b - y).$$

Solution: The problem is represented as shown in Fig. 20.6.

The three possible solutions of the Laplace equation (20.121) are:

I. $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

II. $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

III. $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$.

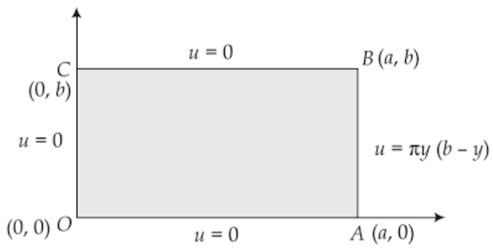


Fig. 20.6

It is easy to see that subject to the given boundary conditions, the solutions II and III lead to trivial solutions. Thus the suitable solution is I which is of the form

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py), \qquad ...(20.122)$$

where A, B, C, D and p are constants to be determined.

Using $u(x, 0) = 0$ in (20.122) gives $0 = (Ae^{px} + Be^{-px})C$, for all $x \in (0, a)$ which implies $C = 0$, and hence (20.122) becomes

$$u(x, y) = D(Ae^{px} + Be^{-px}) \sin py. \qquad ...(20.123)$$

Using $u(x, b) = 0$ in (20.123), gives $0 = D(Ae^{px} + Be^{-px}) \sin pb$ for all $x \in (0, a)$, which implies $\sin pb = 0$ (since D can't be zero), that is, $pb = n\pi$, or $p = \frac{n\pi}{b}$, where n is any integer

Further using $u(0, y)$ in (20.123), we obtain

$0 = D(A + B) \sin py$, for all $y \in (0, b)$ which gives $A + B = 0$, that is $A = B$; and hence (20.123) becomes

$$u_n(x, y) = D_n \sin \frac{n\pi y}{b} \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}} \right).$$

By superposition principle, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} 2D_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}. \qquad ...(20.124)$$

Using the boundary condition $u(a, y) = \pi y(b - y)$ in (20.124), gives

$$\pi y(b - y) = \sum_{n=1}^{\infty} \left(2D_n \sinh \frac{n\pi a}{b} \right) \sin \frac{n\pi y}{b},$$

which is the half-range sine series of $f(y) = \pi y(b - y)$ in the interval $(0, b)$; hence the coefficients $2D_n \sinh \frac{n\pi a}{b}$ are given by

$$2D_n \sinh \frac{n\pi a}{b} = \frac{2\pi}{b} \int_0^b y(b - y) \sin \frac{n\pi y}{b} dy.$$

Consider

$$I = \int_0^b y(b - y) \sin \frac{n\pi y}{b} dy$$

$$= \left[y(b - y)\left(\frac{-b}{n\pi}\right) \cos \frac{n\pi y}{b} - (b - 2y)\left(\frac{-b^2}{n^2\pi^2}\right) \sin \frac{n\pi y}{b} + (-2)\left(\frac{b^3}{n^3\pi^3}\right) \cos \frac{n\pi y}{b} \right]_0^b$$

$$= \frac{2b^3}{n^3\pi^3} [1 - (-1)^n].$$

Hence, $$2D_n \sinh \frac{n\pi a}{b} = \begin{cases} \dfrac{8b^2}{n^3\pi^2}, & \text{when } n \text{ is odd.} \\ 0, & \text{when } n \text{ is even.} \end{cases}$$

Substituting in (20.124), we get

$$u(x, y) = \frac{8b^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3 \sinh \frac{(2n-1)\pi a}{b}} \sin \frac{(2n-1)\pi y}{b} \sinh \frac{(2n-1)\pi x}{b}$$

as the desired solution.

Example 20.13: A long rectangular plate of width π cm with insulated surfaces has its temperature equal to zero on both the long sides and one of the short sides so that $u(0, y) = 0$, $u(\pi, y) = 0$, $u(x, \infty) = 0$, $u(x, 0) = kx$. Find the steady state temperature within the plate.

Solution: The steady state temperature $u(x, y)$ at any point is given by the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \text{...(20.125)}$$

The boundary conditions, as shown in Fig. 20.7 are

$$u(0, y) = 0, \quad \text{for} \quad 0 < y < \infty$$
$$u(\pi, y) = 0, \quad \text{for} \quad 0 < y < \infty$$
$$u(x, \infty) = 0, \quad \text{for} \quad 0 < x < \pi$$
$$u(x, 0) = kx, \quad \text{for} \quad 0 < x < \pi.$$

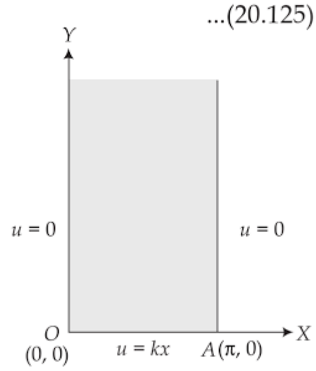


Fig. 20.7

The three possible solutions of the Laplace equation are

I. $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

II. $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

III. $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$.

From the condition that $u = 0$ as y tends to infinity for all values x, we observe that solutions I and III lead to trivial solution and hence solution II is the only suitable one, which is of the form

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \qquad \text{...(20.126)}$$

Using the boundary condition $u(0, y) = 0$, (20.126) gives

$$0 = A\,(Ce^{py} + De^{-py})$$

which implies $A = 0$, and hence (20.126) reduces to the form

$$u(x, y) = \sin px(C'e^{py} + D'e^{-py}). \qquad \text{...(20.127)}$$

Next using the condition $u(\pi, y) = 0$, (20.127) gives $\sin p\pi = 0$, or $p = n$, n being an integer.

Also using $u(x, \infty) = 0$ in (20.127) implies $C' = 0$.

Substituting $p = n$ and $C' = 0$ in (20.127), we obtain $u_n(x, y) = D \sin nxe^{-py}$, n being integer and hence the general solution is of the form

$$u(x, y) = \sum_{n=1}^{\infty} D_n \sin nxe^{-py}. \qquad \text{...(20.128)}$$

Using the boundary condition $u(x, 0) = kx$ in (20.128), we obtain

$$kx = \sum_{n=1}^{\infty} D_n \sin nx, \quad 0 < x < a,$$

which is half-range sine series expansion of $f(x) = kx$ in the interval $[0, \pi]$ hence the coefficients D_n are given by

$$D_n = \frac{2}{\pi}\int_0^{\pi} kx \sin nx\, dx = \frac{2k}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2k}{n}(-1)^{n+1}.$$

Substituting for D_n in (20.128), we get

$$u(x, y) = 2k\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nxe^{-ny},$$

as the desired solution.

EXERCISE 20.4

1. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$, subject to the boundary conditions

$$u(0, y) = u(a, y) = 0, \quad u(x, b) = 0 \quad \text{and} \quad u(x, 0) = f(x).$$

2. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0$, subject to the boundary conditions

$$u(0, y) = u(\pi, y) = 0, \quad u(x, 0) = 1 \text{ and } \quad u(x, y) \to 0 \text{ as } y \to \infty.$$

3. A rectangular plate with insulated surface is 8 cm wide and is infinitely long. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin(\pi x/8)$, $0 < x < 8$, while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plate is given by

$$u(x, y) = 100\, e^{-\pi y/8} \sin(\pi x/8).$$

4. Find the steady state temperature in a rectangular plate when the sides $x = 0$, $x = a$, $y = b$ are insulated while the edge $y = 0$ is kept at temperature $k \cos(\pi x/a)$.

5. A rectangular plate with insulated surface is 10 cm wide and is much long as compared to its width. If the temperature at the short edge $y = 0$ is given by

$$f(x) = \begin{cases} 20x, & \text{for } 0 \le x \le 5 \\ 20(10 - x), & \text{for } 5 \le x \le 10 \end{cases}$$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C. Show that steady state temperature distribution $u(x, y)$ at any point (x, y) in the plate is given by

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{10} e^{-(2n-1)\pi y/100}.$$

6. Find the steady-state temperature distribution in a thin sheet of metal plate which occupies the semi-infinite strip, $0 \le x \le a$ and $0 \le y < \infty$, when the edge $y = 0$ is kept at temperature $f(x) = kx(a - x)$, $0 < x < a$, while

   (i) the edges $x = 0$ and $x = a$ are kept at zero temperature,

   (ii) the edges $x = 0$ and $x = a$ are insulated,

   assuming in both the cases that $u(x, \infty) = 0$.

7. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature $u(x, y)$ at any point (x, y) of the plate. Hence, show that the temperature at the centre of the plate is

$$\frac{200}{\pi}\left[\frac{1}{\cosh(\pi/2)} - \frac{1}{3\cosh(3\pi/2)} + \frac{1}{5\cosh(5\pi/2)} - \ldots\right].$$

8. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $0 < x < a$, $0 < y < b$, subject to the boundary conditions

$$u(0, y) = u(a, y) = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=b} = 0 \quad \text{and} \quad u(x, 0) = f(x).$$

9. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $0 < x < a$, $0 < y < b$, subject to the boundary conditions

   $u(x, 0) = 0$, $u(0, y) = u(a, y) = 0$ and $u(x, b) = (a - x) \sin x$.

10. Solve for the steady-state temperature distribution in a thin, flat plate covering the rectangle $0 \le x \le 4$, $0 \le y \le 1$, if the temperature on the horizontal sides is zero, while on the left side it is $f(y) = \sin \pi y$ and on the right side it is $f(y) = y(1 - y)$.

20.9 SOLUTION OF LAPLACE'S EQUATION IN POLAR COORDINATES

In case we need to solve the steady-state temperature distribution problem for a disc of radius R, then, in general, it is convenient to employ polar coordinates (r, θ) related to (x, y) by $x = r\cos\theta$, $y = r\sin\theta$. The Laplace equation in the cartesian form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is then replaced by its polar form given by

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \qquad \text{...(20.129)}$$

see Appendix 1 (p. 532).

Using the method of separation of variables, let the solution of Eq. (20.129) be of the form

$$u(r, \theta) = R(r)\Theta(\theta), \qquad \text{...(20.130)}$$

where R is a function of r only and Θ that of θ only. Substituting (20.130) in the Eq. (20.129), we get

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0,$$

where primes denote the derivatives w.r.t. the corresponding independent variables.

Separating the variables, we obtain

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta}. \qquad \text{...(20.131)}$$

Since the left side of Eq. (20.131) is a function of r only and right a function of θ only and, r and θ being independent, thus (20.131) holds when each side is equal to a constant, say k. This leads to two differential equations

$$r^2R'' + rR' - kR = 0 \qquad \text{...(20.132)}$$

and,
$$\Theta'' + k\Theta = 0. \qquad \text{...(20.133)}$$

Substituting $r = e^z$ in Eq. (20.132), it reduces to

$$\frac{d^2R}{dz^2} - kR = 0, \qquad \text{...(20.134)}$$

a differential equation with constant coefficients.

To solve Eqs. (20.133) and (20.134), we consider the following cases

(a). When k is positive, say $k = p^2$, then

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p}$$
$$\Theta = c_3 \cos p\theta + c_4 \sin p\theta.$$

(b). When k is negative, say $k = -p^2$, then

$$R = c_5 \cos pz + c_6 \sin pz = c_5 \cos (p \ln r) + c_6 \sin (p \ln r)$$
$$\Theta = c_7e^{p\theta} + c_8e^{-p\theta}.$$

(c). When k is zero, then

$$R = c_9z + c_{10} = c_9 \ln r + c_{10}$$
$$\Theta = c_{11}\theta + c_{12}.$$

Thus the three possible solutions of the Laplace Eq. (20.129) are:

I. $u(r, \theta) = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta)$

II. $u(r, \theta) = [c_5 \cos (p \ln r) + c_6 \sin (p \ln r)](c_7 e^{p\theta} + c_8e^{-p\theta})$

III. $u(r, \theta) = (c_9 \ln r + c_{10})(c_{11}\theta + c_{12})$.

Of these we select the solution(*s*) compatible with the boundary conditions given. In the examples to follow we shall solve the Laplace equation (20.129) subject to the various boundary conditions.

$$A_n a^n = \frac{2}{\pi}\int_0^{\pi} k\theta(\pi-\theta)\sin n\theta\, d\theta$$

$$= \frac{2k}{\pi}\left[\theta(\pi-\theta)\left(-\frac{\cos n\theta}{n}\right)-(\pi-2\theta)\left(-\frac{\sin n\theta}{n^2}\right)+(-2)\left(\frac{\cos n\theta}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4k}{\pi n^3}[1-(-1)^n]$$

or,
$$A_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8k}{\pi a^n n^3}, & \text{when } n \text{ is odd} \end{cases} \quad ...(20.142)$$

Using (20.142) in (20.141), we get

$$u(r,\theta) = \frac{8k}{\pi}\sum_{n=1,3,5}^{\infty}{}' \left(\frac{r}{a}\right)^n \frac{1}{n^3}\sin n\theta$$

$$= \frac{8k}{\pi}\sum_{n=1}^{\infty}{}' \left(\frac{r}{a}\right)^{2n-1} \frac{1}{(2n-1)^3}\sin(2n-1)\theta,$$

as the desired solution.

Example 20.15: Solve the Laplace equation

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \le r \le a, \quad 0 \le \theta \le 2\pi. \quad ...(20.143)$$

subject to the condition

$$u(a,\theta) = f(\theta), \quad -\infty < \theta < \infty. \quad ...(20.144)$$

Solution: The solutions of the Laplace equation (20.143) are

I. $u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_3\cos p\theta + c_4\sin p\theta)$

II. $u(r,\theta) = [c_5\cos(p\ln r) + c_6\sin(p\ln r)][c_7 e^{p\theta} + c_8 e^{-p\theta})$

III. $u(r,\theta) = (c_9\ln r + c_{10})(c_{11}\theta + c_{12})$.

We need to find the solution of Eq. (20.143) subject to the condition (20.144). The *p.d.e.* (20.143) is of order two so we need additional boundary conditions to find the solution.

First reasonable condition we impose is that

$$u(r,\theta) \text{ is finite as } \quad r \to 0, \quad ...(20.145)$$

and, another that $u(r,\theta)$ is the single valued function of θ, and hence

$$u(r,\theta+2\pi) = u(r,\theta). \quad ...(20.146)$$

Thus the possible solution is of the form I. Consider the solution as

$$u(r, \theta) = (Ar^p + Br^{-p})(C \cos p\theta + D \sin p\theta), \qquad ...(20.147)$$

where A, B, C, D and p are constants.

Using condition (20.145) in (20.147), we obtain $B = 0$, thus (20.147) becomes

$$u(r, \theta) = r^p(C' \cos p\theta + D' \sin p\theta). \qquad ...(20.148)$$

Next, we subject (20.148) to the condition (20.146). Obviously $\cos p\theta$ and $\sin p\theta$ are periodic but we are to determine 'p' such that these are 2π-periodic functions.

First we find p such that $\cos p(\theta + 2\pi) = \cos p\theta$, for all θ, which implies

$$\cos p\theta \cos 2\pi p - \sin p\theta \sin 2\pi p = \cos p\theta.$$

Equating the coefficients of $\cos p\theta$ and $\sin p\theta$, we have

$$\left.\begin{aligned} &\cos 2p\pi = 1, \text{ which gives } p = 0, 1, 2, \ldots \\ \text{and,} \quad &\sin 2p\pi = 0, \text{ which gives } p = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \end{aligned}\right\} \qquad ...(20.149)$$

Since both conditions at (20.149) need to be held simultaneously, we select the common values of p and hence

$$p = 0, 1, 2, \ldots$$

Similarly, considering $\sin p(\theta + 2\pi) = \sin p\theta$, for all θ leads to $p = 0, 1, 2, \ldots$

Thus by superposition principle the solution (20.148) can be expressed as

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n(C_n \cos n\theta + D_n \sin n\theta)$$

or,

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n(C_n \cos n\theta + D_n \sin n\theta). \qquad ...(20.150)$$

Using condition (20.144) in (20.150), we obtain

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} a^n(C_n \cos n\theta + D_n \sin n\theta), \; -\infty < \theta < \infty. \qquad ...(20.151)$$

We observe that (20.151) is the Fourier series expansion of the function $f(\theta)$ with period 2π, and hence

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta, \quad C_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta, \text{ and } \quad D_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta \qquad ...(20.152)$$

Thus, the solution of the given boundary value problem (20.143) is (20.150), where C_0, C_n and D_n are given by (20.152). The value of $u(r, \theta)$ as $r \to 0$ is given by C_0.

Example 20.16: Solve the Laplace equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0,\quad 0\le r\le 4,\quad -\pi\le\theta\le\pi, \qquad \text{...(20.153)}$$

subject to the conditions

(i) $u(r, \theta)$ is finite as $r \to 0$,

(ii) $u(r, \theta) = u(r, \theta + 2\pi)$,

(iii) $u(4, \theta) = \theta^2$ for $-\pi \le \theta \le \pi$.

Solution: Proceeding as in Example (20.15), the solution of the Laplace Equation (20.153) subject to the given boundary conditions is

$$u(r,\theta)=A_0+\sum_{n=1}^{\infty} r^n(A_n\cos n\theta+B_n\sin n\theta), \qquad \text{...(20.154)}$$

where $A_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)d\theta,\quad A_n=\dfrac{1}{\pi a^n}\displaystyle\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta,$ and $B_n=\dfrac{1}{\pi a^n}\displaystyle\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta.$

Here, $f(\theta) = \theta^2$ and $a = 4$.

Thus, $$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta^2 d\theta=\frac{\pi^2}{3},$$

$$A_n=\frac{1}{\pi 4^n}\int_{-\pi}^{\pi}\theta^2\cos n\theta\, d\theta=\frac{4(-1)^n}{n^2 4^n}.$$

$$B_n=\frac{1}{\pi 4^n}\int_{-\pi}^{\pi}\theta^2\sin n\theta\, d\theta=0.$$

Substituting for A_0, A_n and B_n in (20.154), the required solution becomes

$$u(r,\theta)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{r}{4}\right)^n\cos n\theta.$$

Example 20.17: Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, $x^2+y^2<9$, subject to the condition

$$u(x, y) = x^2y^2,\ \text{for } x^2 + y^2 = 9.$$

Solution: Using the transformation $x = r\cos\theta$, $y = r\sin\theta$ the boundary value problem becomes

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 3, \qquad \text{...(20.155)}$$

subject to $u(r, \theta) = r^4 \cos^2 \theta \sin^2 \theta = 81 \cos^2 \theta \sin^2 \theta = \frac{81}{4} \sin^2 2\theta = f(\theta)$, say.

The solution of Eq. (20.155) subject to the conditions that $u(r, \theta)$ is finite as $r \to 0$ and $u(r, \theta) = u(r, \theta + 2\pi)$, refer Example (20.15), is

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta), \qquad \text{...(20.156)}$$

where $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta, \quad A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$ and $B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta.$

Here $f(\theta) = \frac{81}{4} \sin^2 2\theta$ and $a = 3$.

Thus, $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{81}{4} \sin^2 2\theta \, d\theta = \frac{81}{8\pi}\left[\int_0^{\pi} (1 - \cos 4\theta) d\theta\right] = \frac{81}{8}.$

$$A_n = \frac{1}{\pi 3^n} \int_{-\pi}^{\pi} \frac{81}{4} \sin^2 2\theta \cos n\theta \, d\theta = \frac{81}{4\pi 3^n} \int_{-\pi}^{\pi} \sin^2 2\theta \cos n\theta \, d\theta.$$

$$= \frac{81}{8\pi 3^n}\left[\int_{-\pi}^{\pi} [\cos n\theta - \cos 4\theta \cos n\theta] \, d\theta\right]$$

Since, $\int_{-\pi}^{\pi} \cos n\theta = 0$ and $\int_{-\pi}^{\pi} \cos 4\theta \cos n\theta \, d\theta = \begin{cases} \pi, \text{ if } n = 4 \\ 0, \text{ otherwise} \end{cases}$

thus, $A_n = -\frac{81}{8.3^n}$, if $n = 4$, otherwise, zero,

Proceeding on similar lines, we obtain

$$B_n = \frac{1}{\pi 3^n} \int_{-\pi}^{\pi} \frac{81}{4} \sin^2 2\theta \sin n\theta \, d\theta = 0, \quad \text{for all } n.$$

Substituting for A_0, A_n and B_n in (20.156), we obtain

$$u(r, \theta) = \frac{81}{8} - \frac{1}{8} r^4 \cos 4\theta, \qquad \text{...(20.157)}$$

as the solution of the boundary value problem in polar coordinates.

To convert (20.157) in terms of cartesian coordinates, we use

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Thus, $$u(r, \theta) = \frac{81}{8} - r^4\cos^4\theta + r^4\cos^2\theta - \frac{r^4}{8}.$$

Substituting, $x = r\cos\theta$ and $r^2 = x^2 + y^2$, it gives

$$u(x, y) = \frac{81}{8} - x^4 + (x^2 + y^2)x^2 - \frac{(x^2+y^2)^2}{8}$$

as the desired solution.

EXERCISE 20.5

1. In a semicircular plate with insulated faces of radius a, bounding diameter at 0°C and the semicircular boundary at 100°C, show that the steady state temperature distribution is given by

$$u(r, \theta) = \frac{400}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n-1}\frac{\sin(2n-1)\theta}{(2n-1)}.$$

2. A semicircular plate of radius 10 cm has insulated faces with bounding diameter at 0°C and circumference temperature distribution is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$.

   Determine the steady state temperature distribution of the plate at any point (r, θ) of the plate.

3. The bounding diameter of a semicircular plate of radius a is kept at 0°C and the temperature along the semicircular boundary is given by

$$u(a, \theta) = \begin{cases} 50\theta, & \text{when } 0 \le \theta < \pi/2 \\ 50(\pi - \theta), & \text{when } \pi/2 \le \theta \le \pi. \end{cases}$$

   Show that the steady state temperature distribution is given by

$$u(r, \theta) = \frac{200}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{r}{a}\right)^{2n-1}\frac{\sin(2n-1)\theta}{(2n-1)^2},$$

   assuming the lateral surfaces of the plate to be insulated.

4. A plate laterally insulated in the shape of truncated quadrant of a circle is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. The plate is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the steady-state temperature distribution.

5. Solve the Laplace equation

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \le r \le b,\ -\infty < \theta < \infty,$$

   subject to the boundary conditions

   (i) $u(r, \theta)$ is finite as $r \to 0$,

21 CHAPTER Descriptive Statistics, Probability and Distributions

"The aspects of collection, processing, analysis and interpretation of data are covered under the domain of statistics. Processing of data is important to study their properties and extract information from them. If data are random in nature they are governed by probability laws and also may follow some specific pattern which makes their study quite systematic and informative."

21.1 BASIC CONCEPTS

Statistics is the art of learning from data. *It is a science which deals with the collection, processing, analysis and interpretation of the numerical data*. Normally, we are interested in obtaining information about the total collection of elements, which we refer to as '*population*' e.g., all the cars produced by a particular company during the last year, or all the students enrolled in an institution this year. We try to learn about the population by selecting and then examining a subgroup of its elements (or members), called a '*sample*'. Population may, or may not be finite in size but sample is always finite.

Variables and Data: *A variable is a characteristic that changes or varies over time, or varies for different individuals or objects under consideration.* For example, body temperature is a variable which changes with time for an individual and also it varies from individual to individual at a particular time.

A variable is measured on an individual or object under consideration called the *experimental unit.* The collection of such measurements form the *data,* and the set of all measurements of interest for every experimental unit in the entire collection form the *population*. Any smaller subset of the measurements forms the *sample*.

For example, say we are interested in the body weights of the trainees registered in a sports academy, then the set of measurements of the body weights of all the trainees, say 500 form the population. If we select say 50 trainees out of the 500 registered by adopting certain methodology, then the set of measurements of the body weights of these selected 50 trainees forms the sample.

If a single variable is measured on a single experimental unit, then data obtained is *univariate data*, e.g., blood pressures of employees working in an organization. When two variables are measured on a single experimental unit then data obtained is *bivariate data*, e.g., blood pressure and

weight of employees. Similarly, if our interest lies in more than two characteristics then data obtained is *multivariate data*.

Categories of variables: Variables can be classified in two categories: *qualitative* and *quantitative*.

A *qualitative variable* measures a quality or characteristic on each experimental unit. For example, colour of the skin: fair, wheatish, black; performance of an individual: excellent, good, fair, average, poor.

A *quantitative variable* measures a quantity or amount on each experimental unit. For example, weight, height, or marks obtained by the experimental unit under observation. Further quantitative variables are of two types: *discrete* and *continuous*. *If a variable can assume only finite or at the most countabley infinite number of values then it is said to be 'discrete'. For example, number of students in a class. However, if a variable can assume infinite number of values between two specific limits, then it is said to 'continuous'*, e.g., time, temperature, weight all are continuous variables. In case of continuous variable, for any two values selected a third value can always be found between the two.

21.2 DATA REPRESENTATION

The numerical findings of a study should be presented in such a systematical manner so that an observer is in a position to grasp the essential characteristics of the data. The data can be represented numerically or graphically in various ways. In this section we present some common graphical and tabular ways for data presentation.

Frequency Tables and Graphs

A data set having a relatively small number of distinct values can be presented in the form of a frequency table. For example, the following frequency table gives the starting salary per month for graduate students of a class of 50 students.

Starting salary (in 000' Rs.)	*Frequency*
40	2
42	3
44	4
45	5
47	5
48	7
49	8
51	7
52	4
54	3
55	2
Total	50

Data can be graphically represented by a line graph by plotting data values along x-axis and corresponding frequencies along y-axis as shown in Fig. 21.1.

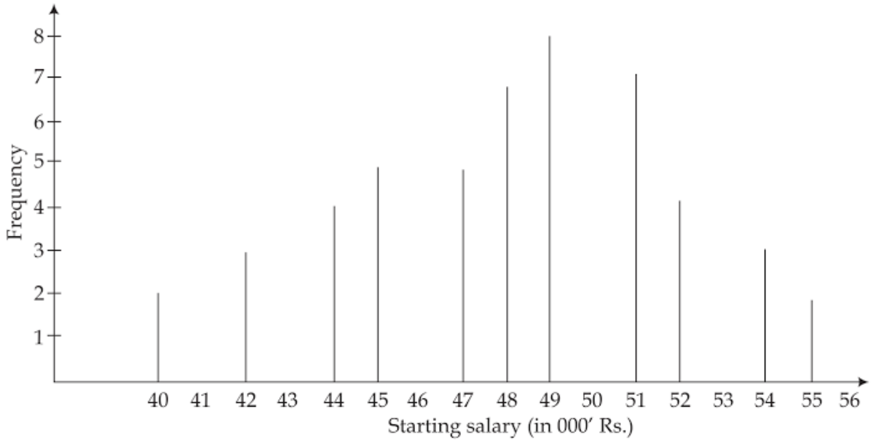


Fig. 21.1

The above frequency distribution is called the *ungrouped frequency distribution*. This frequency distribution, particularly when the data is large can be condensed in the form of the following *grouped distribution*.

Salary-group (class)	*No. of candidates (frequency)*
40 – 44, (40 and under 44)	05
44 – 48	14
48 – 52	22
52 – 56	09
Total	50

This grouped frequency distribution can be represented graphically by *histogram*, *frequency polygon, frequency curve* and *cumulative frequency curve (s)*.

Histogram

A histogram is composed of a set of rectangles, one over each class interval on the horizontal scale. The area of the rectangles are taken proportional to the frequencies of the classes. Thus, in case of equal class-interval, the heights of the rectangles are proportional to the frequencies while for classes of unequal width, the heights are taken proportional to the ratio of the frequencies to the width of the corresponding classes. Figure 21.2 is the histogram for the grouped frequency distribution given above.

The basis of the rectangles in Fig. 21.2 are the class-intervals 40 – 44, 44 – 48, 48 – 52, 52 – 56 whose mid-points (class marks) are x = 42, 46, 50, 54, respectively. Since the class widths are equal, heights of the rectangles are proportional to respective class frequencies 5, 14, 22 and 9. Thus, areas of these rectangles are proportional to these class frequencies.

Fig. 21.2

Frequency Polygon and Frequency Curve

The *frequency polygon* of a grouped frequency distribution is obtained by joining the points whose abscissae are the mid-points, that is, *class marks* of the classes and the ordinate are the corresponding frequencies by means of straight lines. This can be obtained from a histogram by joining the mid-points of the upper sides of the adjacent rectangles by means of straight lines. The points on the horizontal axis at the midpoints of the intervals, immidiately preceeding and immidiatey proceding the intervals that contain observations are also joined, as shown in Fig. 21.2.

If the class-intervals are of small width the frequency polygon can be approximated by a smooth *frequency curve* obtained by drawing a smooth free-hand curve through the vertices of the frequency polygon.

Cumulative Frequency Curve(s), or Ogive

Sometimes we are interested in plotting a cumulative frequency curve, or an ogive. There are two types of cumulative frequencies '*less than*' and '*greater than*'. For example, the grouped frequency distribution for the salary of group of 50 students can be expressed as

Salary (less than)	*Cumulative frequency*
44	05
48	19
52	41
56	50

or, it can be expressed as

Salary (greater than)	*Cumulative frequency*
40	50
44	45
48	31
52	09

$$u = \frac{x-a}{h}, \text{ or } \quad x = a + hu,$$

and thus, $$\bar{x} = \frac{1}{N}\Sigma fx = \frac{1}{N}\Sigma f(a+hu) = a + h\bar{u}.$$

We can choose 'a' and 'h' as any suitable values depending upon the data given. Normally, a is the mid-point of the class-interval corresponding to which frequency is maximum and h is taken as the *H.C.F.* of the widths of the various class-intervals.

Example 21.2: The following data gives the frequencies of serum cholesterol level of 1000 males aged between 25 to 35 years arrived at a particular city hospital during the last one year.

Choslestrol level (mg/100 ml.)	*Number of males*
80 – 120	12
120 – 160	145
160 – 200	380
200 – 240	292
240 – 280	118
280 – 320	35
320 – 360	11
360 – 400	07

Calculate (a) the mean, (b) the median, and (c) the mode for the data.

Solution: (a) To calculate mean take a = 180 and h = 40 and formulate the following table

Class-interval	*Mid-value (x)*	*Frequency (f)*	$u = \frac{x-180}{40}$	*fu*
80 – 120	100	12	–2	–24
120 – 160	140	145	–1	–145
160 – 200	180	380	0	0
200 – 240	220	292	1	292
240 – 280	260	118	2	236
280 – 320	300	35	3	105
320 – 360	340	11	4	44
360 – 400	380	07	5	35
Total		1000		591

The mean, $$\bar{x} = a + h\bar{u} = 180 + 40\left(\frac{591}{1000}\right) = 180 + 23.64 = 203.64 \text{ mg/100 ml.}$$

(b) To calculate median formulate the cumulative frequencies table

Class-interval	Frequency (f)	Cumulative frequency (C)
80 – 120	12	12
120 – 160	145	157
160 – 200	380	537
200 – 240	292	829
240 – 280	118	947
280 – 320	35	982
320 – 360	11	993
360 – 400	07	1000

We have, $\dfrac{N}{2}$ = 500. Thus 160 – 200 is the median class. Therefore,

$$l = 160, \qquad f = 380, \qquad h = 40, \qquad C = 157$$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - C\right) = 160 + \frac{40}{380}(500 - 157) = 160 + 36.11 = 196.11 \text{ mg/100 ml.}$$

(c) Since, the maximum frequency is 380, thus 160 – 200 is the modal class. Therefore,

$$l = 160, \quad f_m = 380, \quad f_1 = 145, \quad f_2 = 292, \quad h = 40.$$

$$\text{Mode} = l + \frac{(f_m - f_1)h}{2f_m - (f_1 + f_2)} = 160 + \frac{(380 - 145)40}{2(380) - (145 + 292)}$$

$$= 160 + 29.10 = 189.10 \text{ mg/100 ml.}$$

Properties of arithmetic mean

We state below a few properties satisfied by the arithmetic mean which can be proved very easily.

1. *Algebraic sum of the deviations of a set of observations from their arithmetic mean is zero, that is, for the frequency distribution* $x_i \mid f_i$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} f_i(x_i - \bar{x}) = 0$, *where* $\bar{x} = \dfrac{1}{N}\sum_{i=1}^{n} f_i x_i$.

2. *The sum of the squares of the deviations of a set of observations is minimum about its mean, that is,* $\sum_{i=1}^{n} f_i(x_i - a)^2$ *is minimum at* $a = \bar{x}$.

3. *The mean of the combined sample is the weighted mean of the individual sample means, that is, if* $\bar{x}_i$ *is the mean of the ith sample of size* n_i, $i = 1, 2, \ldots, k$, *then the mean* $\bar{x}$ *of the combined sample of size* $\sum_{i=1}^{k} n_i$ is,

$$\bar{x} = \sum_{i=1}^{k} n_i \bar{x}_i \bigg/ \sum_{i=1}^{k} n_i .$$

(IV) Geometric mean: *The geometric mean G of a set of N observations is the Nth root of their product, that is, if* $x_i \mid f_i$, $i = 1, 2, \ldots, n$ *is the frequency distribution, then*

$$G = \left(x_1^{f_1} x_2^{f_2} \ldots x_n^{f_n}\right)^{\frac{1}{N}}, \text{ where } \qquad N = \Sigma f_i \qquad \ldots(21.4)$$

This can be written as

$$\ln G = \frac{1}{N}\sum_{i=1}^{n} f_i \ln x_i, \quad \text{or} \quad G = \text{antilog}\left(\frac{1}{N}\sum_{i=1}^{n} f_i \ln x_i\right),$$

an expression normally used for computational purpose.

Obviously it is not practicable to calculate G when one of the values is zero or non-negative.

(V) Harmonic mean: *The harmonic mean H of a set of N non-zero observations is the reciprocal of the arithmetic mean of the reciprocals of the data values, that is, if* $x_i \mid f_i$, $i = 1, 2, \ldots, n$ *is the frequency distribution, then*

$$H = \frac{1}{\frac{1}{N}\sum_{i=1}^{n}(f_i/x_i)}; \quad N = \sum_{i=1}^{n} f_i\,. \qquad \ldots(21.5)$$

There are specific situations when a particular measure of central tendency is appropriate to use. For example, both mean and median each provide a single number to represent an entire set of data, the mean is usually preferred in problems of estimation and statistical inference since it depends upon all the information contained in the data set. The main advantage of median over mean is that it is least affected by the extreme values, for example, median salary is the more representative one than the mean salary in a small company where the top management is highly paid.

Mode is normally most suitable to use to find the ideal size, for example, in manufacturing the readymade garments, shoes, etc.

Geometric mean is used to find the average rate of population growth and construction of index numbers. Harmonic mean attaches greater importance to numerically small observation and hence is useful when situation demands so. In addition to this, in certain situations true average of a data set is given by the harmonic mean of observation values and not by the arithmetic mean. For example, if a motorist travels first 50 km with a speed of 60 km/hr, next 50 km with a speed of 75 km/hr and another 50 km with a speed of 80 km/hr, then his true average speed will be H.M. of 60, 75 and 80 and not the A.M. of 60, 75 and 80. However, barring any particular situation the arithmetic mean is the most suitable measure.

Partition Values: The median value divides a frequency distribution in two equal parts. Partition values divide the distribution into a number of equal parts.

The three values Q_i, $i = 1, 2, 3$ *which divide the distribution in four equal parts are called* ***quartiles****. Obviously* Q_2 *is the median.*

The nine values D_i, $i = 1, 2, \ldots, 9$ *which divide the distribution in ten equal parts are called* ***deciles*** *and the ninety nine values* P_i, $i = 1, 2, \ldots, 99$ *which divide it in hundred equal parts are called* ***percentiles****.*

For a grouped data quartiles can be calculated by using the formula

$$Q_i = l + \frac{h}{f}\left(\frac{iN}{4} - C\right), i = 1, 2, 3. \qquad \ldots(21.6)$$

$$\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = \frac{1}{N}\sum_{i=1}^{n} f_i\,[(a + hu_i) - (a + h\,\bar{u})]^2 = \frac{h^2}{N}\sum_{i=1}^{n} f_i(u_i - \bar{u})^2,$$

or, $$\sigma_x^2 = h^2\,\sigma_u^2 \qquad \text{...(21.9)}$$

which gives $$\sigma_x = |h|\,\sigma_x\,. \qquad \text{...(21.10)}$$

Thus, there is no effect of change of origin on the variance but there is effect of change of scale on the variance as given by (21.9).

Also consider

$$\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = \sum_{i=1}^{n} f_i(x_i^2 + \bar{x}^2 - 2x_i\,\bar{x}) = \sum_{i=1}^{n} f_i x_i^2 + N\bar{x}^2 - 2N\,\bar{x}^2 = \sum_{i=1}^{n} f_i x_i^2 - N\,\bar{x}^2.$$

Thus, the variance given by (21.7) can be written as

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i\,x_i^2 - \left(\frac{1}{N}\Sigma\, f_i x_i\right)^2. \qquad \text{...(21.11)}$$

We use the expression (21.11) for computation of the variance, since it reduces the calculations to a great extent.

Example 21.3: Following data give the daily emission (in tonnes) of sulphur oxides from an industrial plant observed for 80 days. Calculate the mean, variance and S.D. of the daily emission.

Emission (in tonnes)	*No. of days*
5 – 9	3
9 – 13	10
13 – 17	14
17 – 21	25
21 – 25	17
25 – 29	9
29 – 33	2
Total	80

Solution: We take $a = 19$ and $h = 4$ and formulate the following table:

Class interval	*Mid-point (x)*	*Frequency (f)*	$u = \dfrac{x-19}{4}$	$f_i u_i$	$f_i u_i^2$
5 – 9	7	3	– 3	– 9	27
9 – 13	11	10	– 2	– 20	40
13 – 17	15	14	– 1	– 14	14
17 – 21	19	25	0	0	0
21 – 25	23	17	1	17	17
25 – 29	27	9	2	18	36
29 – 33	31	2	3	6	18
Total		80		– 2	152

Thus, the mean

$$\bar{x} = a + h\bar{u}$$

$$= 19 + 4\left(\frac{-2}{80}\right) = 18.90 \text{ tonnes}$$

The variance of u's

$$\sigma_u^2 = \frac{1}{N}\Sigma f_i u_i^2 - \left(\frac{1}{N}\Sigma f_i u_i\right)^2$$

$$= \frac{152}{80} - \left(\frac{-2}{80}\right)^2 = 1.9 - .000625 = 1.899375$$

Hence $\sigma_x^2 = 16\ \sigma_u^2 = 30.39$ sq. tonnes.

Thus, S.D. $\sigma_x = 5.51$ tonnes.

(IV) Coefficient of variation: It does not make any sense to compare the variability of the two frequency distributions which differ widely in their averages or measured in different units. In such a situation we use *coefficient of variation (C.V.), which is the ratio of the standard deviation σ to the mean $\bar{x}$ of a data set multiplied by 100. It is a measure of relative variability and is a dimensionless number.* The C.V. for a data set is given by

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100. \qquad ...(21.12)$$

The ratio $\sigma/\bar{x}$ is called the *coefficient of dispersion based upon S.D.*

Example 21.4: The following data gives the weight and chest size of 10 infants at birth in a city hospital. Compare the variability of the two characteristics in the infants

Weight (in kg)	*Chest size (in cm.)*
2.75	29.1
3.12	30.1
4.15	32.1
5.50	36.1
3.20	30.2
4.32	33.1
2.31	28.2
5.12	35.1
4.12	31.9
3.72	31.1

Solution: We formulate the following table.

	Weight (x)	Chest size (y)	x^2	y^2
	2.75	29.1	7.56	846.81
	3.12	30.1	9.73	906.01
	4.15	32.1	16.93	1030.41
	5.50	36.1	30.25	1303.21
	3.20	30.2	10.24	912.04
	4.32	33.1	18.66	1095.61
	2.31	28.2	5.34	795.24
	5.12	35.1	26.21	1232.01
	4.12	31.9	16.97	1017.61
	3.72	31.1	13.83	967.21
Total	38.31	317.0	155.72	10106.16

We have, $n = 10, \quad \bar{x} = \frac{1}{n} \Sigma x_i = 3.83, \quad \bar{y} = \frac{1}{n} \Sigma y_i = 31.7$

$$\sigma_x^2 = \frac{1}{n} \Sigma x_i^2 - (\bar{x})^2 = 15.57 - 14.67 = 0.9$$

$$\sigma_y^2 = \frac{1}{n} \Sigma y_i^2 - (\bar{y})^2 = 1010.61 - 1004.89 = 5.72$$

Thus, $\text{C.V.}(x) = \frac{\sigma_x}{\bar{x}} \times 100 = 24.77$, and $\text{C.V.}(y) = \frac{\sigma_y}{\bar{y}} \times 100 = 7.54$.

Since C.V. for infant weight is greater than the C.V. for infant chest size, thus infant weight is more variable than infant chest size.

21.3.3 Moments

The rth moment of a variable x about any point x = a, also called the 'ordinary moment', denoted by μ_r', is defined by

$$\mu_r' = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a)^r; \quad N = \sum_{i=1}^{n} f_i. \qquad ...(21.13)$$

In case $a = \bar{x}$, then the rth moment about mean, also called the 'central moment' denoted by μ_r, is defined by

$$\mu_r = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^r \qquad ...(21.14)$$

In particular, we have $\mu_0 = 1, \quad \mu_1 = 0, \quad \mu_2 = \sigma^2$.

Also $\mu_0' = 1, \quad \mu_1' = \bar{x} - a$. Thus if $a = 0$, then $\mu_1' = \bar{x}$, hence the *first moment about the origin is the mean.*

Relation between μ_r and μ'_r. We have

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r = \frac{1}{N}\sum_i f_i[(x_i - a) - (\bar{x} - a)]^r$$

$$= \frac{1}{N}\sum_i f_i\left[C_0^r(x_i - a)^r - C_1^r(x_i - a)^{r-1}(\bar{x} - a) + C_2^r(x_i - a)^{r-2}(\bar{x} - a)^2 + \ldots + (-1)^r C_r^r(\bar{x} - a)^r\right]$$

$$= \mu'_r - C_1^r \mu'_{r-1}\mu'_1 + C_2^r \mu'_{r-2}(\mu'_1)^2 + \ldots + (-1)^r(\mu'_1)^r \qquad \ldots(21.15)$$

In particular, for $r = 2, 3, 4$ we have

$$\left.\begin{aligned} \mu_2 &= \mu'_2 - (\mu'_1)^2 \\ \mu_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 \\ \mu_4 &= \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4 \end{aligned}\right\} \qquad \ldots(21.16)$$

These formulae give us moments about mean once the moments about any arbitrary point 'a' are known. Since moments about mean are particularly employed to know the nature of the frequency distribution, hence the above noted formulae are quite useful.

We can check very easily that *there is no effect of change of origin on the central moments but there is effect of change of scale. The rth moment of the variable x about mean is h^r times the rth moment of the variable u about mean when $u = (x - a)/h$.*

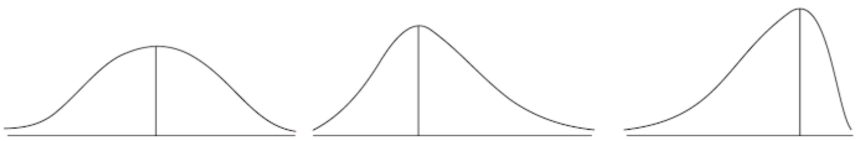
Pearson's β and γ coefficients. The four important coefficients are defined as follow.

$$\left.\begin{aligned} \beta_1 &= \frac{\mu_3^2}{\mu_2^3}, \quad \gamma_1 = \sqrt{\beta_1} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2}, \quad \gamma_2 = \beta_2 - 3 \end{aligned}\right\} \qquad \ldots(21.17)$$

These four coefficients are pure numbers, that is, they are independent of origin and scale of the measurement of the variate and are widely used to know the nature of the frequency distribution.

21.3.4 Skewness and Kurtosis

Skewness means *lack of symmetry. A distribution is 'symmetric', or 'normal' when the frequencies are symmetrically distributed about mean*, refer Fig. 21.4a. Any data set which is not approximately



Mean = Mode = Median
(symmetric)
(a)

Mode < Median < Mean
(Positively skewed)
(b)

Mean < Median < Mode
(Negatively skewed)
(c)

Fig. 21.4

(b) Construct a histogram.

(c) Draw cumulative frequency curves 'less than' and 'greater than' both.

2. The following data gives the ages of the 40 coins in circulation, where

Age = Current year – Year on coin

5	1	9	1	2	25	0	25
1	4	4	3	0	25	3	3
5	21	19	9	0	5	0	2
0	1	19	0	2	0	20	16
19	36	23	0	1	17	6	0

Draw a histogram to describe the distribution of coin ages. How do you describe the shape of the distribution?

3. Determine the mean, median and mode for the following data values:

(a) 3, 10, 8, 7, 5, 14, 2, 9, 8

(b) 73.8, 126.4, 40.7, 141.7, 28.5, 237.4, 157.9

4. The gain of 90 similar transistors is measured and the results are recorded as given below:

Gain:	83.5- 85.5	86.5-88.5	89.5-91.5	92.5-94.5	95.5-97.5
No. of transistors:	6	39	27	15	3

Determine the mean, median and the modal values of the distribution.

5. An incomplete frequency distribution is given as follows.

Class-interval	*Frequency*	*Class-interval*	*Frequency*
10 – 20	12	50 – 60	?
20 – 30	30	60 – 70	25
30 – 40	?	70 – 80	18
40 – 50	65		
		Total	229

If the median value is 46, determine the missing frequencies.

6. The nicotine content in milligrams for 40 cigarettes of a certain brand are given as follows:

1.09	1.92	2.31	1.79	2.28
1.74	1.47	1.97	0.85	1.24
1.58	2.03	1.70	2.17	2.55
2.11	1.86	1.90	1.68	1.51
1.64	0.72	1.69	1.85	1.82
1.79	2.46	1.88	2.08	1.67
1.37	1.93	1.40	1.64	2.09
1.75	1.63	2.37	1.75	1.69

Find the mean, median and S.D.

7. Find the mean and variance of the first n natural numbers.

total number of possible outcomes of a random experiment is known as *exhaustive events*. For example, in tossing a coin the set $\{H, T\}$ forms the set of exhaustive cases while, in case of throw of a dice the exhaustive set is $\{1, 2, 3, 4, 5, 6\}$.

Mutually exclusive events: *Events are called mutually exclusive, if the occurrence of one of them precludes the occurrence of all the others*. For example, in tossing a coin the two outcomes head and tail are mutually exclusive.

Equally like events: *Events are called equally likely when we have no reason to expect one in preference to the others.* For example, as the result of drawing a card from a well shuffled pack any card out of the 52 cards may appear as the result of the draw.

Favourable cases: *Cases which entail or favour the happening of an event A are said to be favourable cases to the event A*. For example, if A is an event that in a throw of a dice face obtained is less than 4, then 1, 2, 3 forms a set of three favourable cases to the event A.

Now we are in a position to discuss the concept of probability.

21.4.1 Classical Probability Concept

If a trial may result in one of the 'n' exhaustive, mutually exclusive and equally likely cases out of which m are favourable to the happening of an event A, then the probability 'p' that the event A will happen as the result of the trial, is given by

$$p = P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{m}{n}$$

Clearly it follows from this definition that $0 \le P(A) \le 1$ and also, if $\overline{A}$, or A^c is the event '*not happening*' of A, then

$$q = P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A),$$

so that $$P(A) + P(\overline{A}) = 1.$$

An event A with $P(A) = 0$ is called an *impossible event* and an event A with $P(A) = 1$ is called a *sure event*.

Example 21.7: From a pack of 52 cards two are drawn at random. Find the chance that one is a king and the other a queen.

Solution: Let A be the event that the two cards drawn are a king and a queen.

$$\text{No. of exhaustive cases} = {}^{52}C_2 = \frac{52!}{50!\,2!} = 26 \times 51 = 1326$$

No. of cases favourable to $A = 4 \times 4$.

Therefore, $$P(A) = \frac{16}{1326} = .01207.$$

Example 21.8: A five-figure number is formed by the numbers 0, 1, 2, 3, 4, without repetition. Find the probability that number formed is divisible by 4.

Solution: The five digits can be arranged in 5! ways and out of these 4! will begin with zero. Thus, number of five digits formed, that is, exhaustive cases are

$$= 5! - 4! = 96.$$

A number will be divisible by 4 which will have two extreme right digits divisible by 4, that is, the number ending with 04, 12, 20, 24, 32 and 40.

Numbers ending with 04 = 3! = 6
Numbers ending with 12 = 3! – 2! = 4
Numbers ending with 20 = 3! = 6
Numbers ending with 24 = 3! – 2! = 4
Numbers ending with 32 = 3! – 2! = 4
Numbers ending with 40 = 3! = 6.

Thus total numbers divisible by 4, that is, favourable cases are

$$= 6 + 4 + 6 + 4 + 4 + 6 = 30.$$

Hence the required probability $= \dfrac{30}{96} = 0.3125.$

Example 21.9: If 5 of 20 fuses in a box are defective and 5 of them are randomly chosen for inspection, what is the probability that two of the defective fuses will be included?

Solution: Let A be event that out of 5 selected, two will be defective and three will be non-defective.

$$\text{Exhaustive cases} = {}^{20}C_5 = \frac{20!}{15!\,5!} = 15504$$

$$\text{Cases favourable to } A = C_2^5 \times C_3^{15} = \frac{5!}{3!\,2!} \times \frac{15!}{12!\,3!} = 4550$$

Therefore, $$P(A) = \frac{4550}{15504} = 0.293.$$

Example 21.10: A committee of 4 members is to be appointed from 3 officers of the production department, 4 officers of the purchase department, 2 officers of the sales department and 1 Chartered Accountant. Find the probability of forming the committee in the following manners

(a) There must be one from each category,
(b) It should have at least one from the purchase department,
(c) The Chartered Accountant must be in the committee.

Solution: Total number of persons out of which four members are to be selected

$$= 3 + 4 + 2 + 1 = 10.$$

$$\text{Exhaustive number of cases} = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

(a) Favourable number of cases for committee to consist of one member from each category

$$= C_1^3 \times C_1^4 \times C_1^2 \times C_1^1 = 3 \times 4 \times 2 \times 1 = 24.$$

Therefore desired probability $= \dfrac{24}{210} = 0.114$ (approx.).

(b) Let A be the event that committee has at least one purchase officer, then A^c is event that committee has no purchase officer.

$$\text{Cases favourable to } A^c = C_4^6 = \frac{6 \times 5}{2 \times 1} = 15$$

Therefore, $$P(A^c) = \frac{15}{210} = 0.071 \text{ (approx).}$$

Hence, $P(A) = 1 - P(A^c) = 0.929$ (approx).

(c) Favourable number of cases to include one Chartered Accountant out of 4 are

$$1 \times C_3^9 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

Hence, the desired probability $= \dfrac{84}{210} = 0.40.$

Example 21.11: A class in probability theory consists of 6 boys and 4 girls. An examination is conducted and the students are ranked according to their performance. Assume that no two students obtain the same score, what is the probability that girls receive the top 4 scores?

Solution: Since each ranking corresponds to a particular ordered arrangement of the 10 students, thus total number of different rankings = 10!.

Since there are 4! possible rankings among the girl students and 6! possible rankings among the boy students, so the total number of ways in which the four girls can receive the top rankings

$$= 4! \times 6!$$

Hence the desired probability $= \dfrac{4! \times 6!}{10!} = \dfrac{4 \times 3 \times 2 \times 1}{10 \times 9 \times 8 \times 7} = \dfrac{1}{210}$.

Example 21.12: From a set of n items a random sample of size k is to be selected. What is the probability a given item will be among the k selected?

Solution: Number of exhaustive cases $= C_k^n$.

Number of different selections that contain the given item $= C_1^1 \times C_{k-1}^{n-1} = C_{k-1}^{n-1}$

Hence, the desired probability $= C_{k-1}^{n-1} \big/ C_k^n = \dfrac{(n-1)!}{(k-1)!\,(n-k)!} \times \dfrac{(n-k)!}{n!} k! = \dfrac{k}{n}$.

Major shortcomings of the classical concept is that it is applicable only to equally likely possibilities. In addition to this, it is applicable only when the number of exhaustive cases are finite; and also it gives probability of an event only as a rational number in [0, 1]. There are numerous situations in which various possibilities cannot be regarded as equally likely, e.g., if we are concerned with the question whether it will rain tomorrow, whether the flight will have the safe landing under a particular weather condition, etc., then the different possibilities are not equally likely and so the classical concept fails in such situations.

Another concept of probability, which is most widely used is the frequency interpretation concept, called the *statistical,* or *empirical probability concept,* as discussed next.

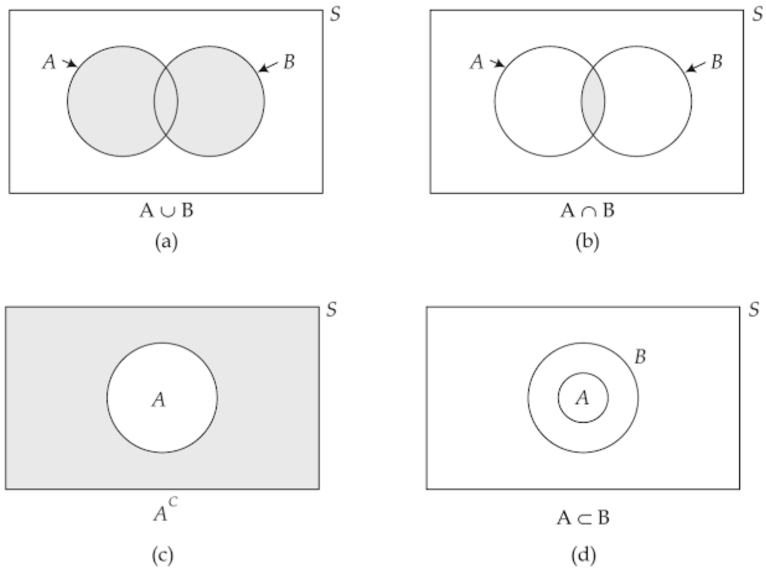


Fig. 21.6

Axiom 2: *$P(S) = 1$*

Axiom 3: *If A and B are mutually exclusive events in S, then*

$$P(A \cup B) = P(A) + P(B).$$

We can show that the axiomatic probability concept is in consistent with the classical and empirical concepts of the probability. On the basis of the axiomatic approach the theory of probability is developed which forms the basis of statistical inference.

Remarks:

1. Using mathematical induction, Axiom 3 can be extended to any number of mutually exclusive events $A_1, A_2, ..., A_n$ in S, that is,

$$P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n). \qquad \ldots(21.18)$$

2. *The probability of the impossible event ϕ is zero*. Since, $S \cup \phi = S$ and $S \cap \phi = \phi$, thus from Axiom 3, $P(S) + P(\phi) = P(S)$ which implies $P(\phi) = 0$.

3. *If A is in S, then $P(A^c) = 1 - P(A)$*. Since, $A \cup A^c = S$ and $A \cap A^c = \phi$, thus from Axiom 3, $P(A) + P(A^c) = P(S)$, which implies $P(A^c) = 1 - P(A)$.

The results **2** and **3** are in conformity with the results already studied.

4. The *odds of an event A* is defined by $P(A)/P(A^c) = P(A)/(1 - P(A))$. It tells how much more likely it is that A occurs than that it does not occur. If $P(A) = 2/5$, then $P(A)/(1 - P(A)) = \frac{2}{5}\Big/\frac{3}{5} = 2/3$. So the odds is $2/3$, hence it is $2/3$ times as likely that A occurs as it is that it does not, that is $P(A): P(A^c) ::$ $2/3 : 1$, or $2 : 3$.

Example 21.13: If an experiment has four possible and mutually exclusive outcomes A, B, C and D specify in the following cases whether the assignment of probability is permissible.

(a) $P(A) = 1/3,\quad P(B) = 1/6,\quad P(C) = 1/4,\quad P(D) = 1/4.$

(b) $P(A) = 1/4,\quad P(B) = 1/6,\quad P(C) = 1/3,\quad P(D) = 1/3.$

Solution: (a) All the assignments of probabilities are in the interval [0, 1] and $P(A) + P(B) + P(C) + P(D) = 1$. Hence the assignments are permissible.

(b) Assignments lie in the interval [0, 1], but $P(A) + P(B) + P(C) + P(D) = 13/12 > 1$. Hence the assignments are not permissible.

Example 21.14: A, B, and C are three mutually exclusive and exhaustive events associated with a random experiment. Find $P(A)$, if $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$.

Solution: Since, A, B and C are mutually exclusive and exhaustive events, thus

$$P(A) + P(B) + P(C) = 1.$$

or, $$P(A) + \frac{3}{2}P(A) + \frac{1}{2}\left(\frac{3}{2}P(A)\right) = 1 \quad \text{or,} \quad \frac{13}{4}P(A) = 1 \quad \text{or,} \quad P(A) = 4/13.$$

21.5 ADDITION AND MULTIPLICATION LAWS OF PROBABILITY

In this section, we consider two basic laws of probability, addition law and multiplication law.

21.5.1 Addition Law of Probability, or Theorem of Total Probability

Theorem 21.1 (Addition law): *If A and B are any two events in the sample space S, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \qquad ...(21.19)$$

Proof. From Fig. (21.7), we have

$$A \cup B = A \cup (A^c \cap B),$$

where A and $A^c \cap B$ are mutually exclusive.

Therefore, by Axiom 3

$$P(A \cup B) = P(A) + P(A^c \cap B) \qquad ...(21.20)$$

Also from Fig. 21.7

$$B = (A \cap B) \cup (A^c \cap B),$$

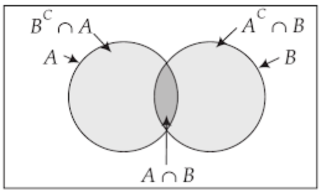


Fig. 21.7

and again $A \cap B$ and $A^c \cap B$ are mutually exclusive. Therefore,

$$P(B) = P(A \cap B) + P(A^c \cap B).$$

or, $$P(A^c \cap B) = P(B) - P(A \cap B). \qquad ...(21.21)$$

Substituting for $P(A^c \cap B)$ in (21.20), we obtain (21.19).

In case the events A and B are mutually exclusive, then $A \cap B = \phi$ and hence (21.19) gives

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom 3.

The result (21.19) can be extended to more than two events, say for three events A, B and C in S, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C), \quad \text{...(21.22)}$$

and so on.

Example 21.15: A total of 21 per cent male employees of a company smoke cigarettes, 5 per cent smoke cigar and 3 percent smoke both cigar and cigarette. What percentage of males smokes neither cigar nor cigarette?

Solution: Let a male employee of the company is selected at random and A be the event that selected individual smokes cigarette and B be the events that he smokes cigar. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.21 + 0.05 - 0.03 = 0.23.$$

Thus 0.23 is the probability that employee is a smoker. Hence 1 – 0.23 = 0.77 is the probability of employee being non-smoker. Thus 77% employees are non-smokers.

Example 21.16: In tossing an unbiased dice, what is the probability of getting an odd number or a number less than 4?

Solution: Let A be the event 'number is odd' and B the event 'number is less than 4'. Then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}.$$

Example 21.17: What is the probability of getting a total of 7 or 11, when a pair of unbiased dice are tossed?

Solution: Let A be the event that 'total is 7' and B be the event that 'total is 11.'

No. of exhaustive cases = 36.

Number of cases favourable to A = 6

Number of cases favourable to B = 2

Thus, $P(A) = 6/36 = 1/6$ and $P(B) = 2/36 = 1/18$.

Further, A and B are mutually exclusive, therefore,

$$P(A \cup B) = P(A) + P(B) = 1/6 + 1/18 = 2/9.$$

Example 21.18: A graduate student applies for a job in two companies X and Y. The probability of being selected in X is 0.6 and being rejected in Y is 0.4. The probability of at least one of his applications being rejected is 0.5. What is the probability of getting job?

Solution: Let A be the event getting job in X and B be the event getting job in Y, then $A \cup B$ is the event getting job.

Here $\quad P(A) = 0.6, \quad P(\overline{B}) = 0.4, \quad P(\overline{A} \cup \overline{B}) = 0.5,$

We have, $\quad P(\overline{A} \cup \overline{B}) = P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cap \overline{B}).$

It gives $\quad P(\overline{A} \cap \overline{B}) = P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cup \overline{B}) = 0.4 + 0.4 - 0.5 = 0.3$

Thus, $\quad P(A \cup B) = 1 - P(\overline{A} \cap \overline{B}) = 1 - 0.3 = 0.7.$

Example 21.19: A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution: Let A be the event 'getting a king', B be the event 'getting a heart' and, C be the event 'getting a red card'.

Obviously A, B, C are not mutually disjoint events. We are interested in the event $A \cup B \cup C$. We have,

$$P(A) = 4/52 = 1/13 \quad P(B) = 13/52 = 1/4 \quad P(C) = 26/52 = 1/2$$

$$P(A \cap B) = 1/52 \qquad P(B \cap C) = P(B) = 1/4$$

$$P(C \cap A) = 2/52 \quad P(A \cap B \cap C) = P(A \cap B) = 1/52.$$

Thus,
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{1}{13} + \frac{1}{4} + \frac{1}{2} - \frac{1}{52} - \frac{1}{4} - \frac{2}{52} + \frac{1}{52} = \frac{1}{13} + \frac{1}{2} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}.$$

Compound events. Independent events. Conditional probability

When two or more events occur in connection with each other, their simultaneous occurrence is called a 'compound event'.

Events are said to be 'independent', if the probability of the occurrence of one does not depend on the occurrence or non-occurrence of the others, otherwise, the events are said to be 'dependent'.

For example, let A be the event that first draw from a pack of 52 cards is queen and B be the event that second draw is a king. Then $P(A) = 4/52 = 1/13$ and $P(B) = 3/51$ or $4/51$ depending upon whether the first draw was 'a king' or 'not a king'. Hence, A and B are dependent events.

In case the second card has been drawn after replacing the first, then $P(B) = 1/13$. It does not depend upon whether the first draw was a king or not. Hence A and B are independent events in this case.

The probability for the event A to occur, when it is known that the event B has already occurred is called the 'conditional probability of A given B' and is denoted by $P(A \mid B)$.

As an example, consider that a pair of unbiased dice are tossed. Then there are 36 possible outcomes given by $S = \{(i, j) : i = 1, 2, ..., 6, j = 1, 2, ..., 6\}$.

Let A be the event that 'sum of the two dice equals 8', then the cases favourable to A are: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) and hence $P(A) = 5/36$.

Let B be the event that 'face of the first dice is 3'. Then A/B is the event that 'sum of the two dice equals 8 when the face of the first dice is 3'.

Exhaustive cases are (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6) and favourable case is (3, 5). Hence, the conditional probability of A given B denoted by $P(A \mid B) = 1/6$.

We observe that $P(A \mid B) \neq P(A)$ hence the event A is dependent on B.

If $P(A \mid B) = P(A)$, then event A is called independent of the event B.

(a) the balls are returned to the bag after the first draw.
(b) the balls are not returned.

Solution: (a) The number of ways in which two balls out of 7 may be drawn = $\overset{7}{C}_2$.

The number of ways in which 2 red balls out of 4 may be drawn = $\overset{4}{C}_2$.

Thus, the probability of drawing two red balls = $\overset{4}{C}_2 \big/ \overset{7}{C}_2 = \dfrac{4!}{2!\,2!} \times \dfrac{5!\,2!}{7!} = \dfrac{2}{7}$.

Similarly, the probability of drawing two blue balls at the second draw = 2/7.

Therefore the desired probability = $\dfrac{2}{7} \times \dfrac{2}{7} = \dfrac{4}{49}$.

(b) As in (a) the probability of drawing two red balls at the first draw = 2/7.

Since the balls are not returned, the probability of drawing two blue balls at the second draw

$$= \overset{3}{C}_2 \big/ \overset{5}{C}_2 = 3/10.$$

Therefore, the desired probability in this case = $\dfrac{2}{7} \times \dfrac{3}{10} = \dfrac{3}{35}$.

Example 21.21: Mr. X works out that there is 50 per cent chance that his company will set up a branch office in New Delhi. If it is so, he is 80 per cent certain that he will be assigned the responsibilities of manager of the new set up. What is the probability that Mr. X will be a New Delhi branch office manager?

Solution: Let D be the event that branch office will be set up at New Delhi and M the event that Mr. X will be made manager there. Then

$$P(DM) = P(D)\,P(M \mid D) = \frac{1}{2} \times \frac{4}{5} = \frac{2}{5}.$$

Hence there are 40% chance that Mr. X will be a New Delhi branch officer manager.

Example 21.22: Two cards are drawn from a pack of 52 cards. Find the probability that draw includes an ace and a ten.

Solution: Let the event A: Draw an ace and a ten. Then $A = B \cup C$, where

B : First draw an ace and second draw a ten

C : First draw a ten and second draw an ace

Now, $P(B) = \dfrac{4}{52} \times \dfrac{4}{51}$ and $P(C) = \dfrac{4}{52} \times \dfrac{4}{51}$.

Also we observe that B and C are mutually exclusive events, thus applying addition rule

$$P(A) = P(B) + P(C) = \frac{32}{52 \times 51} = \frac{8}{663}.$$

Example 21.23: A system composed of k separate components is said to be a parallel system if it functions when at least one of the k component functions. For such a system, if p_i, independent of others, is the probability that ith component will function $i = 1, 2, ..., k$, then what is the probability that system will function?

Solution: Let A_i be the event that the ith component functions. Then,

$$\begin{aligned} P[\text{System functions}] &= 1 - P[\text{System does not function}] \\ &= 1 - P[\text{No component functions}] \\ &= 1 - P[A_1^c A_2^c \ldots A_k^c] \\ &= 1 - \prod_{i=1}^{k} (1 - p_i). \end{aligned}$$

Example 21.24: A system consists of four components as shown in Fig. 21.8. System functions if components A and B both function and at least one of the components C or D functions. If the probabilities of functioning components A, B, C and D, respectively are 0.8, 0.8, 0.6 and 0.6, find the probability that, (a) entire system functions and, (b) the component C does not function given that the system functions. Assume that the components function independently.

Solution: (a) Let A be the event that component A functions, and so on. Then the event that entire system functions is $A \cap B \cap (C \cup D)$.

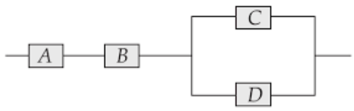


Fig. 21.8

Therefore,

$$\begin{aligned} P(A \cap B \cap (C \cup D)) &= P(A)P(B)P(C \cup D) = P(A)P(B)[1 - P(\overline{C} \cap \overline{D})] \\ &= P(A)\,P(B)\,[1 - P(\overline{C})\,P(\overline{D})] = (0.8)\,(0.8)\,[1 - (1 - 0.6)\,(1 - 0.6)] \\ &= (0.64)\,(.84) = 0.5376. \end{aligned}$$

(b) $P[C \text{ does not function} \mid \text{System functions}]$

$$= \frac{P[\text{ System functions and } C \text{ does not function }]}{P[\text{System functions}]}$$

$$= \frac{P[A \cap B \cap \overline{C} \cap D]}{P[\text{System functions}]} = \frac{P(A)P(B)P(\overline{C})P(D)}{P[\text{System functions}]}$$

$$= \frac{(0.8)(0.8)(0.4)(0.6)}{0.5376} = 0.2857.$$

Example 21.25: The odds that a research monograph will be accepted by 3 independent referees are 3 to 2, 4 to 3 and 2 to 3, respectively. Find the probability that of the three reports,

(a) all will be favourable,
(b) majority of the reports will be favourable,
(c) at least one of the reports will be favourable.

Solution: Let A, B and C be the events that monograph is accepted favourably by the referee I, II and III, respectively. Then

$$P(A) = 3/5, \quad P(B) = 4/7, \quad P(C) = 2/5,$$
$$P(\bar{A}) = 2/5, \quad P(\bar{B}) = 3/7, \quad P(\bar{C}) = 3/5.$$

(a) $A \cap B \cap C$ is the event all will be favourable.

$$P(A \cap B \cap C) = P(A)P(B)P(C) = 3/5 \times 4/7 \times 2/5 = \frac{24}{175}.$$

(b) The event that majority, that is, at least two will be favourable happens when (i) $A \cap B \cap \bar{C}$, or (ii) $A \cap \bar{B} \cap C$, or (iii) $\bar{A} \cap B \cap C$, or (iv) $A \cap B \cap C$ happens; and all these are mutually exclusive.

Hence the desired probability $= P(AB\bar{C}) + P(A\bar{B}C) + P(\bar{A}BC) + P(ABC)$

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{2}{5} + \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5}$$

$$= \frac{36}{175} + \frac{18}{175} + \frac{16}{175} + \frac{24}{175} = \frac{94}{175}.$$

(c) $\quad P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$$= 1 - \frac{2}{5} \times \frac{3}{7} \times \frac{3}{5} = 1 - \frac{18}{175} = \frac{157}{175}.$$

Example 21.26: Suppose an assembly plant receives its voltage regulators from three different sources, 60% from B_1, 30% from B_2 and 10% from B_3. Let 95%, 80% and 65% of the supply received respectively from the sources B_1, B_2 and B_3 perform as per specifications laid. If A is the event that a voltage regular received at the plant performs as per specification, then find $P(A)$.

Solution: We can express

$$A = A \cap (B_1 \cup B_2 \cup B_3) = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3).$$

Since B_1, B_2, B_3 are mutually exclusive, therefore, $(A \cap B_1)$, $(A \cap B_2)$ and $(A \cap B_3)$ are also so, and hence

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) = P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)$$
$$= (0.60)(0.95) + (0.30)(0.80) + (0.10)(0.65) = 0.57 + 0.24 + 0.065 = 0.875,$$

is the probability that any one voltage regulator received at the company will perform as per specifications laid.

21.6 BAYES' RULE

Let us suppose we want to extend the forgoing problem as discussed in Example (21.26). We want to know the probability that a particular voltage regulator which is performing as per specifications came from some specific source, say B_1. Thus, we want to know $P(B_1 \mid A)$. We have

$$P(B_1 \mid A) = \frac{P(A \cap B_1)}{P(A)}, \quad P(A) > 0.$$

$$= \frac{P(B_1)P(A|B_1)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{(0.60)(0.95)}{0.875} = 0.651,$$

using result from Example (21.26).

We observe that the probability that a voltage regulator is supplied by B_1 increases from 0.60 to 0.651 once it is known that it is performing as per specifications.

This method can be extended to yield the result called Bayes' rule as stated below.

Theorem 21.3 (Bayes' rule): *If $B_1, B_2, ..., B_n$ are mutually exclusive events in the sample space S of which one must occur and $P(B_i) \neq 0$, for i = 1, 2, ...k, then for any event A in S such that $P(A) \neq 0$,*

$$P(B_i | A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}, \quad \textit{for} = 1, 2, ..., k. \qquad ...(21.27)$$

Proof. We have, $A = \bigcup_{i=1}^{k} (A \cap B_i)$, where the events $A \cap B_i$, $i = 1, 2, ..., k$, are mutually exclusive.

Hence by addition and multiplication laws of probability

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(B_i)P(A | B_i),$$

Also, $\quad P(A \cap B_i) = P(A)P(B_i | A)$

Hence, $\quad P(B_i | A) = \dfrac{P(A \cap B_i)}{P(A)} = \dfrac{P(B_i)P(A|B_i)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}, \; i = 1, 2, ... k.$

Remark: The probabilities $P(B_i)$ are known as the *'a prior probabilities'* and the probabilities $P(B_i | \text{A})$ are known as the *'posterior probabilities'*.

Example 21.27: The probabilities of X, Y and Z becoming managers of a company are 4/9, 2/9 and 1/3, respectively. The probabilities that the Bonus Scheme will be introduced if X, Y and Z becomes managers are 3/10, 1/2 and 4/5, respectively.

(a) Find the probability that Bonus Scheme will be introduced.

(b) If the Bonus Scheme has been introduced find the probability that manager appointed was X or Y.

Solution: Let B_1, B_2, B_3 be the events that respectively X, Y and Z become manager, and A the event that Bonus Scheme is introduced. We have

$P(B_1) = 4/9, \qquad P(B_2) = 2/9, \qquad P(B_3) = 1/3,$

$P(A | B_1) = 3/10, \qquad P(A | B_2) = 1/2, \qquad P(A | B_3) = 4/5$

(a) We have, $A = \bigcup_{i=1}^{3} (A \cap B_i)$, where $A \cap B_i$, $i = 1, 2, 3$ are mutually exclusive, thus

$$P(A) = \sum_{i=1}^{3} P(B_i)\, P(A \mid B_i) = \frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{1}{2} + \frac{1}{3} \times \frac{4}{5} = \frac{2}{15} + \frac{1}{9} + \frac{4}{15} = \frac{23}{45}.$$

(b) $B_1 \cup B_2$ is the event that manager appointed was X_1 or Y, also $B_1 \cap B_2 = \phi$. Thus,

$$P(B_1 \cup B_2 \mid A) = P(B_1 \mid A) + P(B_2 \mid A)$$

$$= \frac{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)}{P(A)} = \frac{\frac{2}{15} + \frac{1}{9}}{\frac{23}{45}} = \frac{22}{90} \times \frac{45}{23} = \frac{11}{23}.$$

EXERCISE 21.2

1. Two balls are randomly drawn from a bowl containing 6 white and 5 black balls. What is the probability that one of the drawn balls is white and the other black?
2. If n people are present in a room what is the probability that no two of them celebrate their birthday on the same day of the year. (Ignore the possibility of someone having been born on 29th of Feb.) How large need n be, so that, this probability is less than 1/2 ?
3. A dice is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.
4. In a class of 100 students, 54 studied mathematics, 69 studied physics, and 35 studied both mathematics and physics. If one of these students is selected at random, find the probability that
   (a) the student took mathematics or physics,
   (b) the student did not take either of these subjects,
   (c) the student took physics but not mathematics.
5. One shot is fired from each of the three guns. G_1, G_2 and G_3 denote the events that target is hit by the first, second and third gun, respectively. If $P(G_1) = 0.5$, $P(G_2) = 0.6$ and $P(G_3) = 0.8$ and G_1, G_2, G_3 are independent events, find the probability that (a) exactly one hit is registered (b) at least two hits are registered.
6. The odds that X speaks the truth are 3:2 and the odds that Y speaks the truth are 5:3. What is the probability that they are likely to contradict each other on an identical point?
7. A and B cast each with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. Find their respective chances of winning, if A begins.
8. A smoke-detector system uses two devices A and B. If smoke is present the probability that it will be detected by device A is 0.95; by device B, 0.98 and by both devices, 0.94. If smoke is present find the probability that it will be get detected.
9. Six cards are drawn with replacement from an ordinary pack. What is the probability that each of the four suits will be represented at least once among the six cards?

In case of discrete random variable, the relation between probability mass function and distribution function is

$$F(x) = \sum_{i: x_i \le x} p(x_i) \qquad \text{...(21.31)}$$

and, in case of continuous random variable, the relation is

$$F(x) = P\{X \in (-\infty, x]\} = \int_{-\infty}^{x} f(x)dx. \qquad \text{...(21.32)}$$

Differentiating (21.32) w.r.t. x, we have

$$f(x) = \frac{dF(x)}{dx}. \qquad \text{...(21.33)}$$

Also we observe that

$$P(a \le X \le b) = F(b) - F(a) = \int_{a}^{b} f(x)dx; \qquad \text{...(21.34)}$$

and, $\qquad F(-\infty) = 0$ and $F(\infty) = 1$.

In many situations we are interested in relationship between two random variables X and Y. If X and Y are both discrete random variables, then their *joint probability mass function* is defined by

$$p(x, y) = P\{X = x, Y = y\}; \; \sum_{x} \sum_{y} p(x, y) = 1, \qquad \text{...(21.35)}$$

and *joint distribution function* by

$$F(x, y) = P\{X \le x, Y \le y\}. \qquad \text{...(21.36)}$$

The individual distribution functions are obtained by using

$$F_X(x) = \lim_{y \to \infty} F(x, y) \text{ and } F_Y(y) = \lim_{x \to \infty} F(x, y). \qquad \text{...(21.37)}$$

The two discrete random variables X and Y are said to be independent, provided

$$P\{X = x, Y = y\} = P\{X = x\}P\{Y = y\}$$

for all x and y. Also we have in this case

$$F(x, y) = F_X(x)F_Y(y) \qquad \text{...(21.38)}$$

for all x and y.

Similarly, we can define joint distribution in case of continuous random variables X and Y.

Example 21.28: A consignment of 10 similar PCs contains 4 defective PCs. If an institution makes a random purchase of 3 PCs from this consignment, find the probability distribution for the number of defective PCs purchased and the distribution function.

Solution: Let X be the number of defective PCs purchased, then X can take the values $x = 0, 1, 2, 3$. We have

$$p(0) = P(X = 0) = \frac{C_0^4 \times C_3^6}{C_3^{10}} = \frac{20}{120} = 1/6$$

$$p(1) = P(X = 1) = \frac{C_1^4 \times C_2^6}{C_3^{10}} = \frac{60}{120} = 1/2.$$

$$p(2) = P(X = 2) = \frac{C_2^4 \times C_1^6}{C_3^{10}} = \frac{36}{120} = 3/10$$

$$p(3) = P(X = 3) = \frac{C_3^4 \times C_0^6}{C_3^{10}} = \frac{4}{120} = 1/30.$$

Thus, the probability distribution $p(x)$ of X and distribution function $F(x)$ are given by

x :	0	1	2	3
$p(x)$:	1/6	1/2	3/10	1/30
$F(x)$:	1/6	2/3	29/30	1.

Example 21.29: If the life of a component X has the probability density function

$$f(x) = \begin{cases} 2e^{-2x}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$$

find the probabilities that it will take on a value, (a) between 1 and 3, (b) greater than 0.5, (c) find the distribution function.

Solution: (a) $P\{1 < X < 3\} = \int_1^3 2e^{-2x}dx = e^{-2} - e^{-6} = 0.133.$

(b) $P\{X > 0.5\} = \int_{0.5}^{\infty} 2e^{-2x}dx = e^{-1} = 0.368.$

(c) Performing the necessary integration, we obtain the distribution function as

$$F(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x 2e^{-2x}\,dx = 1 - e^{-2x}, & x > 0 \end{cases}$$

Example 21.30: Suppose that 3 batteries are randomly chosen from a group of 3 new and 4 used but still working, and 5 defective batteries. If X and Y denote respectively the number of new and used but still working batteries chosen, then find the joint probability mass function of X and Y.

Solution: Let $p_{ij} = P\{X = i, Y = j\}$, $i, j = 0, 1, 2, 3$ be the probability that out of 3 selected i are new and j are used but still working. Then

$$p_{ij} = \frac{c_i^3 \times c_j^4 \times c_{3-(i+j)}^5}{c_3^{12}}, \quad i, j = 0, 1, 2, 3, \; i + j = 3.$$

We find that

$p_{00} = 10/220$ $\quad p_{01} = 40/220$ $\quad p_{02} = 30/220$ $\quad p_{03} = 4/220$

$p_{10} = 30/220$ $\quad p_{11} = 60/220$ $\quad p_{12} = 18/220$ $\quad p_{13} = 0$

$p_{20} = 15/220$ $\quad p_{21} = 12/220$ $\quad p_{22} = p_{23} = 0$

$p_{30} = 1/220$ $\quad p_{31} = p_{32} = p_{33} = 0.$

The distribution can be conveniently put in the form of the following two-way table given as below

$$[p_{ij} = P_r\{X = i, Y = j\}]$$

$i \backslash j$	0	1	2	3	$P\{x = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
$P\{y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	1

Remarks:

1: The distribution

X :	0	1	2	3
$p(x)$:	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

is called the *marginal distribution* of X ; and the distribution

$Y =$	0	1	2	3
$p(y) =$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$

is called the *marginal distribution* of Y.

Example 21.35: A secretary has typed n letters along with their respective envelopes. The envelopes get mixed up when they fall on the floor. If the letters are placed in the mixed-up envelopes in a completely random manner, what is the expected number of letters that are placed in the correct envelopes?

Solution: Let X denote the number of letters that are placed in the correct envelope, we compute X by considering

$$X = \sum_{i=1}^{n} X_i \text{, where}$$

$$X_i = \begin{cases} 1, & \text{if the } i\text{th letter is in its proper envelope.} \\ 0, & \text{otherwise} \end{cases}$$

We note that $P[X_i = 1] = 1/n$.

Hence, $$E(X_i) = 1.P[X_i = 1] + 0.P[X_i = 0] = \frac{1}{n}.$$

Therefore, $$E(X) = \sum_{i=1}^{n} E(X_i) = n.\frac{1}{n} = 1.$$

Hence irrespective of the number of letters, on the average exactly one letter will be in its proper envelope.

21.8.2 Moment Generating Function

The moment generating function (m. g. f.) of a r.v. X about origin, denoted by $M_0(t)$, is defined by

$$M_o(t) = E[e^{tX}] = \begin{cases} \Sigma\, e^{tx} p(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, & \text{if } X \text{ is continuous.} \end{cases} \qquad \ldots(21.45)$$

$M_o(t)$ is called the m.g.f. because all the moments of X can be obtained from $M_o(t)$, for

$$M_o(t) = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots + \frac{t^rX^r}{r!} + \ldots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots$$

and we observe that

$$\mu_r' = \frac{d^r}{dt^r} M_0(t)\Big|_{t=0} \qquad \text{...(21.46)}$$

The moment generating function of the r.v. X about an arbitrary point 'a', ('a' may be $\overline{X}$ also), is defined by

$$M_a(t) = E[e^{t(X-a)}] = e^{-at}E(e^{tX}).$$

An important property of moment generating functions is that the moment generating function of the sum of two independent random variables is the product of their moment generating functions. Also the moment generating function determines a distribution uniquely.

If X is a r.v which takes only non-negative integral values 0, 1, 2, …, then the expression

$$P_X(t) = E(t^X) = \sum_{x=0}^{\infty} p_i t^x = p_0 + p_1 t + p_2 t^2 + \ldots$$

is called the *probability generating function* (p.g.f.) of X.

Example 21.36: Find the m.g.f. of the exponential distribution $f(x) = ae^{-ax}$, $0 \le x < \infty$, $a > 0$. Hence, find its mean and S.D.

Solution: By definition

$$M_0(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} a e^{-ax} dx = a\int_0^{\infty} e^{(t-a)x} dx,\ t < a.$$

$$= \left(1 - \frac{t}{a}\right)^{-1} = 1 + \frac{t}{a} + \frac{t^2}{a^2} + \frac{t^3}{a^3} + \ldots$$

Therefore, $\mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \frac{1}{a}$, and $\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \frac{2}{a^2}$

Hence, mean $= \frac{1}{a}$ and S.D. $= \sqrt{\mu_2' - (\mu_1')^2} = \sqrt{\frac{2}{a^2} - \frac{1}{a^2}} = \frac{1}{a}$.

Example 21.37: Find the m.g.f. of a Bernoulli variate X with probability mass function $P\{x = 1\} = p$, $P\{x = 0\} = q$, $p + q = 1$. Hence find its mean and variance.

Solution: By definition $M_0(t) = E(e^{tX}) = \Sigma\, e^{tx_i} p(x_i) = pe^t + q$.

Therefore, $\mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = p$, and $\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = p$.

Hence, mean $= p$, and variance $= p - p^2 = pq$.

21.9 CHEBYSHEV'S INEQUALITY

The variance and hence standard deviation of a random variable gives us idea about the variability of the observations about mean. If σ is large there is correspondingly higher probability of getting values farther away from the mean. The Chebyshev's inequality gives us bound on probability that how far a random variable is deviated when both mean and variance of the distribution are known. The result is helpful when the actual distribution of X is not known. It is stated as follows.

Theorem 21.4 (Chebyshev's inequality): If X is a random variable with mean μ and variance σ^2, then for any value $k > 0$

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}. \qquad ...(21.47)$$

Proof. We prove it for the case when X is continuous with density function $f(x)$.

For a non-negative r.v.X,

$$E(X) = \int_0^\infty xf(x)dx = \int_0^a xf(x)dx + \int_a^\infty xf(x)dx, \text{ for any } a > 0$$

$$\geq \int_a^\infty xf(x)dx \geq a\int_a^\infty f(x)dx = aP\{X \geq a\}.$$

Thus, $$P\{X \geq a\} \leq \frac{E(X)}{a}. \qquad ...(21.48)$$

Replacing X by $(X - \mu)^2$ and a by k^2, (21.48) gives

$$P\{(X - \mu)^2 \geq k^2\} \leq \frac{E(X - \mu)^2}{k^2}$$

or, $$p\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2},$$

the Chebyshev's inequality.

Remarks:

1. By replacing k by $k\sigma$ in (21.47), Chebyshev's Inequality can be expressed as

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}. \qquad ...(21.49).$$

2. Sometimes result is applicable in the form

$$P\{|X - \mu| < k\sigma\} \geq 1 - \frac{1}{k^2}. \qquad ...(21.50)$$

3. The inequality (21.48) is known as *Markov inequality* and is of practical importance.

Example 21.38: The number of items cleared by an assembly line during a week is a random variable with mean 50 and variance 25. (a) What is the probability that this week items cleared will exceed 75? (b) What can be said about the probability that this week's clearance will be between 40 to 60?

Solution: Let X be the r.v. denoting the number of items cleared in a week. We have

$$E(X) = 50 \text{ and } \operatorname{Var}(X) = 25.$$

(a) By (21.48), $P\{X > a\} \le \dfrac{E(X)}{a}$, $a > 0$; thus, $P\{x > 75\} \le \dfrac{50}{75} = \dfrac{2}{3}$.

(b) Chebyshev's inequality (21.47) is $P\{|X - \mu| \ge k\} \le \dfrac{\sigma^2}{k^2}$.

Take $k = 10$, it gives, $P\{|X - 50| \ge 10\} \le \dfrac{25}{100} = \dfrac{1}{4}$

or, $P\{|X - 50| < 10\} \ge \dfrac{3}{4}$, which gives $P\{40 < X < 60\} \ge \dfrac{3}{4} = 0.75$.

Example 21.39: Number of customers who visit a car dealer's showroom on weekend is a random variable with mean 18 and S.D. 2.5. What can be said about the probability that on a weekend the customers will be between 8 to 28?

Solution: Let X be the number of customers visiting on weekend, then

$$E(X) = 18 \text{ and } \operatorname{var}(X) = (2.5)^2 = 6.25.$$

Chebyshev's inequality is $P\{|X - \mu| \ge k\} \le \dfrac{\sigma^2}{k^2}$.

Take $k = 10$, the inequality gives, $P\{|X - 18| \ge 10\} \le \dfrac{6.25}{100} = \dfrac{1}{16}$

or, $P\{|X - 18| < 10\} \ge \dfrac{15}{16}$, or $P\{8 < X < 28\} \ge \dfrac{15}{16} = 0.9375$.

EXERCISE 21.3

1. A fair dice is tossed once, find the probability function and distribution function for X, the number obtained.
2. In tossing a fair coin, obtain the probability function for the number of trials until the first head appears.
3. In a lottery 8000 tickets are to be sold at Rs. 5 each. The prize is a Rs. 12,000 T.V. If two tickets are purchased what is the expected gain?
4. A Rs. 5,000 item can be insured for its total value by paying an yearly premium of Rs. N. If the probability of theft in a given year is estimated to be .01, what premium should the insurance company charge if it wants the expected gain to equal Rs. 1000?

13. Over the range of cylindrical parts manufactured on a computer-controlled lathe, the S.D. of the diameters is 0.002 millimetres. What about the probability that a new part will be within 0.006 units of the mean μ for that run? If 400 parts are made during the run, about what portion do you expect will lie in the interval found?
14. A r.v. X with unknown probability distribution has a mean $\mu = 8$ and S.D. $\sigma = 3$. Find, (a) $P\{-4 < X < 20\}$, (*b*) $P\{|X - 8| \geq 6\}$.
15. Show that for 40,000 tosses of a balanced coin, the probability is at least 0.99 that proportion of heads will fall between 0.475 and 0.525.

21.10 SPECIAL DISCRETE PROBABILITY DISTRIBUTIONS

In this section, we consider some special discrete probability distributions which occur frequently in applications. We shall study uniform, binomial, hypergeometric, Poisson, geometric and multinomial distributions.

21.10.1 Discrete Uniform Distribution

If a r.v. X assumes the values $x_1, x_2, \ldots, x_k$ with equal probability, then the random variable is said to follow 'uniform distribution' with probability distribution

$$P\{X = x_i\} = \frac{1}{k}, \quad i = 1, 2, \ldots, k. \qquad \ldots(21.51)$$

Thus for a throw of an unbiased die or random draw of a card from a pack, this distribution is the suitable one. We can easily calculate that in this case

$$E(X) = \frac{k+1}{2}, \qquad E(X^2) = \frac{(k+1)(2k+1)}{6} \quad \text{and} \quad V(X) = \frac{(k+1)(k-1)}{12}$$

21.10.2 Binomial Distribution

First we define *Bernoulli trials*.

Repeated independent trials in which there are only two possible outcomes say 'success' or 'failure' and the probability of success remains constant throughout the trials, are called 'Bernoulli trials'.

For example, repeated tosses of a coin, and say, falling head is classified as 'success' and falling tail as 'failure'. Repeated draws of a card from a pack with replacement and classifying 'success' as the event getting a card of heart on a draw, otherwise 'failure'.

Next, we derive binomial distribution.

Consider a set of n independent Bernoulli trials in which the probability of success is 'p' and of failure is $q = 1 - p$. We wish to find the probability of x successes in n such trials.

First consider the probability of x successes and $(n - x)$ failures in a specified order. It is obviously $p^x q^{n-x}$. Next, these x successes in n trials can occur in C_x^n ways and all these are mutually exclusive. Hence, the requisit probability is $C_x^n p^x q^{n-x}$.

The probability distribution of the number of successes so obtained is called the *Binomial probability distribution*, and the r.v. X giving the number of successes is called the *binomial variate*. Thus,

a manufactured product classified as being good, average, not acceptable is a multinomial distribution.

In general, if a given trial can result in one of the k outcomes $E_1, E_2, ..., E_k$ with probabilities $p_1, p_2, ..., p_k$; $\sum_{i=1}^{k} p_i = 1$ and, if $X_1, X_2, ..., X_k$ are the r.v representing the number of occurrences of $E_1, E_2, ..., E_k$, then the probability that in n trials $X_1, X_2, ..., X_k$ take respectively the values $x_1, x_2, ..., x_k$, is

$$p(x_1, x_2, \ldots, x_k; p_1, p_2, \ldots, p_k, n) = \frac{n!}{x_1! x_2! \ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}, \qquad ...(21.59)$$

where $\sum_{i=1}^{k} x_i = n, \; 0 \le x_i \le n.$

The distribution defined by (21.59) is called *multinomial distribution*. It defines a probability distribution, since

$$\Sigma\, p(x_1, x_2, ..., x_k; p_1, p_2, ..., p_k, n) = \sum \frac{n!}{x_1! x_2! \ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$
$$= (p_1 + p_2 + ... + p_k)^n = 1.$$

Example 21.45: Out of a lot containing 5 good, 4 faulty and 3 partially faulty but working batteries, three have been selected at random with replacement. Find the probability that selection consists of exactly one of each type.

Solution: Let p, q and r be the probabilities of selecting good, faulty and partially faulty batteries at a single draw, then $p = 5/12$, $q = 4/12$, $r = 3/12$.

If X_1, X_2 and X_3 be the r.v. giving the number of good, faulty and partially faulty batteries out of 3, then

$$P_r\{X_1 = 1, X_2 = 1, X_3 = 1\} = \frac{3!}{1!1!1!}\left(\frac{5}{12}\right)\left(\frac{4}{12}\right)\left(\frac{3}{12}\right) = \frac{5}{24}.$$

21.10.4 Hypergeometric Distribution

The difference between binomial distribution and hypergeometric distribution lies in the procedure sampling is made. When the population is finite and sampling is done without replacement so the events although random, become stochastically dependent, then the resulting distribution is no more binomial. It leads to hypergeometric distribution. Let us suppose that a random sample of n items is selected without replacement from N items, k of which are classified as successes and $N - k$ as failures. If X is the r. v giving the number of successes out of the n selected, then

$$P_r\{X = x\} = \frac{C_x^k \times C_{n-x}^{N-k}}{C_n^N}, \; x = 1, 2, ..., \min(n; k) \qquad ...(21.60)$$

The r.v. X defined so is called hypergeometric variable and the distribution defined by (21.60) is called hypergeometric probability distribution. We can see easily that

$$\sum_x P_r\{X = x\} = \frac{\sum_x C_x^k \times C_{n-x}^{N-k}}{N_{C_n}} = N_{C_n}/N_{C_n} = 1.$$

Also, we can show that

$$E(X) = \frac{nk}{N} \quad \text{and} \quad \text{Var}(X) = \frac{Nk(N-k)(N-n)}{N^2(N-1)} \qquad \text{...(21.61)}$$

Hypergeometric distribution tends to binomial distribution when $N \to \infty$ and $\frac{k}{N} = p$, in fact, in this case the sampling becomes equivalent to sampling with replacement.

Applications of the hypergeometric distribution are found in electronic testing and quality assurance when the item tested is destroyed and can't be replaced in the sample.

Example 21.46: A lot consisting of 100 fuses is inspected by the following procedure. Five of these fuses are chosen at random and tested; if 4 or more work at the correct amperage, the lot is accepted. If there are 20 defective fuses in the lot, find the probability of acceptance.

Solution: Let X be the number of fuses working out of the five selected, then

$$P\{X \geq 4\} = P\{X = 4\} + P\{X = 5\}$$

$$= \frac{C_4^{80} \times C_1^{20}}{C_5^{100}} + \frac{C_5^{80} \times C_0^{20}}{C_5^{100}}$$

$$= 0.42 + 0.32 = 0.74.$$

21.10.5 Geometric Distribution

In a Bernoulli sequence of trials with probability of success p, let the *r.v.X* denote the number of failures preceding the first success, then

$$P_r\{X = x\} = q^x p, \quad x = 0, 1, 2, ...; \quad q = 1 - p. \qquad \text{...(21.62)}$$

The r.v X is called *geometric variable* and the distribution defined by (21.62) is called *geometric distribution*. It defines a probability distribution, since

$$\sum_x P_r\{X = x\} = \sum_{x=0}^{\infty} q^x p = p(1 + q + q^2 + ...) = \frac{p}{1-q} = \frac{p}{p} = 1.$$

We can calculate very easily that

$$E(X) = \frac{q}{p}, \quad \text{Var}(X) = \frac{q}{p^2} \quad \text{and} \quad M_0(t) = \frac{p}{1 - qe^t}. \qquad \text{...(21.63)}$$

Example 21.47: If the probability is 0.10 that a certain kind of measuring device will show excessive drift. What is the probability that the fifth measuring device tested will be the first to show excessive drift?

Solution: We have, $p = 0.10$ and $q = 0.90$, thus

$$Pr\{X = 4\} = (0.9)^4(0.10) = 0.066$$

21.10.6 The Poisson Distribution

Consider the situation when in binomial distribution n is large and p is small such that average number of successes np is a finite constant, say equal to λ.

The probability of x successes is given by

$$p(x) = C_x^n p^x q^{n-x}.$$

Rewriting it as

$$p(x) = \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \cdot \frac{n!}{n^x (n-x)!}$$

$$= \frac{\lambda^x}{x!}\left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{x-1}{n}\right).$$

Taking limit as $n \to \infty$, it gives

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \; x = 0, 1, 2, \ldots \infty. \qquad \ldots(21.64)$$

A r.v. X, taking non-negative integral values, with probability distribution (21.64) is called 'Poisson variate' and the distribution is called 'Poisson distribution'. It defines a probability distribution, since

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} \cdot e^{\lambda} = 1.$$

Besides approximating binomial distribution for large n and small p (normally we apply Poisson distribution when $n \geq 20$ and $p \leq 0.05$), the Poisson distribution has numerous applications. A few situations where Poisson variate is applied are:

1. The number of printing errors per page in a printed book.
2. The number of defective fuses in a pack of 100
3. The number of accidents per year at a busy crossing.
4. The number of wrong telephone numbers dialed in a day.
5. The number of α particles discharged in a fixed period of time from some radioactive material.
6. The number of customers arriving at a service counter on a given day.

Constants of a Poisson variate

The *mean*, $$E(x) = \sum_{x=0}^{\infty} x \,.\, e^{-\lambda} \frac{\lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \lambda e^{-\lambda} e^{\lambda} = \lambda. \qquad ...(21.65)$$

Similarly, $$E(x^2) = E[x(x-1) + x] = E[x(x-1)] + E(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)\,.\,e^{-\lambda} \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} x.e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \lambda^2 \sum_{x=2}^{\infty} e^{-\lambda} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \sum_{x=1}^{\infty} e^{-\lambda} \frac{\lambda^{x-1}}{(x-1)!} = \lambda^2 + \lambda$$

Hence the *variance*, $$\text{Var}(x) = E(x^2) - (E(x))^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda. \qquad ...(21.66)$$

Thus in case of Poisson variate mean is equal to variance.

The *m.g.f about origin* is given by

$$M_0(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}\,.\,e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}.\, e^{\lambda e^t} = \exp\,[\lambda(e^t - 1)]. \qquad ...(21.67)$$

The *m.g.f. about mean* is given by

$$M_{\bar{x}}(t) = e^{-\lambda t} M_0(t) = e^{-\lambda t}.e^{-\lambda}.\, e^{\lambda e^t}$$
$$= \exp\,[\lambda(e^t - (1 + t))]. \qquad ...(21.68)$$

From this we can calculate the moments about the mean. We can very easily check that

$$\mu_2 = \frac{d^2}{dt^2}[M_{\bar{x}}(t)]_{t=0} = \lambda,\ \mu_3 = \frac{d^3}{dt^3}[M_{\bar{x}}(t)]_{t=0} = \lambda, \text{ and } \mu_4 = \frac{d^4}{dt^4}[M_{\bar{x}}(t)]_{t=0} = 3\lambda^2 + \lambda \qquad ...(21.69)$$

Thus the *coefficients of skewness and kurtosis* are given by

and, $$\left.\begin{aligned} \beta_1 &= \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda} \end{aligned}\right\} . \qquad ...(21.70)$$

Since $\beta_1 > 0$, so the Poisson distribution is a positively skewed distribution, and also since $\beta_2 > 3$, the distribution is leptokurtic. As $\lambda \to \infty$, then $\beta_1 = 0$ and $\beta_2 = 3$, the distribution tends to be symmetric.

It is easy to verify that the *mode* of Poisson variate is integral part of λ, when λ is not an integer. In case λ is an integer, then λ and $\lambda - 1$ both are the mode, thus distribution is *bimodal*.

Example 21.48: Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2% of such fuses are defective.

Solution: Here $\lambda = np = 200(.02) = 4$.

Hence, the requisite probability is

$$P(x \le 5) = \sum_{x=0}^{5} e^{-4}\frac{4^x}{x!} = e^{-4}\left[1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right]$$

$$= (0.0183)\,[1 + 4 + 8 + 10.6667 + 10.6667 + 8.5333]$$

$$= (0.0183)\,(42.8667) = 0.7845$$

Example 21.49: Consider an experiment that consists of counting the number of α particles given off in a one-second interval by one gram of radioactive material. If past experience shows that on the average 3.2 such α-particles are given off, find the probability that no more than 2 α-particles will appear?

Solution: If r.v X denotes the number of α-particles given off in a second interval, then X will be a Poisson variate with mean $\lambda = 3.2$.

Hence, the requisite probability is

$$P[X \le 2] = \sum_{x=0}^{2} e^{-3.2}\frac{(3.2)^x}{x!} = e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2!}e^{-3.2} = 0.041 + 0.130 + 0.209 = 0.38.$$

Example 21.50: If the average number of road accidents reported daily in a township is 5, what proportion of days have less than 3 accidents reported? What is the probability that there will be 4 accidents reported per day in exactly three days out of five, assuming that the number accidents on different days is independent?

Solution: Let X be the number of accidents reported daily, then X is Poisson variate with mean 5. Hence the probability that there will be less than three accidents reported is

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) = e^{-5} + e^{-5}\frac{5^1}{1!} + e^{-5}\frac{5}{2!} \approx 0.125.$$

Thus over the long run on about 12.5% days the number of accidents reported will be less than or equal to 3.

Since, it has been assumed that number of accidents on different days is independent, thus number of days in a five-day duration that has exactly 4 accidents reported is a binomial distribution with parameter $n = 5$ and probability of 'success' p, given by

$$p = P[X = 4] = e^{-5}\frac{5^4}{4!} \approx 0.175.$$

Thus, the probability that exact 3 out of the next five days will report 4 accidents daily

$$= C_3^5 (0.175)^3 (0.825)^2 = 0.0365.$$

Example 21.51: A manufacturer who produces medicine bottles finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes. Using Poisson distribution find how many boxes will contain (a) no defective, (b) at least two defectives.

Solution: We have,

$N = 100, \quad n = 500, \quad p =$ Probability of a defective bottle $= 0.001$,

$\lambda = np = 500 \times 0.001 = 0.5.$

If the r.v. X denotes the number of defective bottles in a pack of 500, then by Poisson distribution

$$P\{X = x\} = e^{-0.5} \frac{(0.5)^x}{x!} = \frac{0.6065 (0.5)^x}{x!}; \quad x = 0, 1, 2, \ldots$$

Hence in a lot of 100 boxes the frequency of boxes with x defective bottles is given by

$$f(x) = NP\{X = x\} = \frac{100 \times 0.6065 \times (0.5)^x}{x!}$$

(i) Number of boxes with no defective

$$= 100P(X = 0) = 100 \times 0.6065 \approx 61.$$

(ii) Number of boxes with at least two defectives

$$= 100\,[P(X \geq 2)] = 100\,[1 - P(X = 0) - P(X = 1)]$$
$$= 100\,[1 - 0.6065 - 0.6065 \times 0.5] = 100 \times 0.0903 \approx 9.$$

Example 21.52: Fit a Poisson distribution to the following data which gives the number of yeast cells per square for 400 squares

No. of cells per square (x) :	0	1	2	3	4	5	6	7	8	9	10
No. of squares (f) :	103	143	98	42	8	4	2	0	0	0	0

Solution: The parameter λ of the Poisson distribution is given by

$$\lambda = \frac{1}{N} \sum f_i x_i = \frac{529}{400} = 1.32.$$

If the r.v. X denotes the number of yeast cells per square then expected frequencies on the basis of Poisson distribution are given by

$$f(x) = NP\{X = x\} = 400\, e^{-1.32} \frac{(1.32)^x}{x!}, \; x = 0, 1, 2, \ldots, 10.$$

It gives the following frequencies:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	107	141	93	41	14	4	1	0	0	0	0

EXERCISE 21.4

1. It is known that disks produced by a certain company will be defective with probability .01 independently of each other. The company sells the disk in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective. What proportion of packages is returned. If someone buys three packages, what is the probability that exactly one of them will be returned?
2. Over a long period of time it has been observed that a given shooter can hit a target on a single trial with probability equal to 0.8. Suppose he fires four shots at the target. (a) What is the probability that he will hit the target exactly two times? (b) What is the probability that he will hit the target at least once?
3. The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?
4. If the probability that a fluorescent light has a useful life of at least 800 hours is 0.9, find the probabilities that among 20 such lights
   (a) exactly 18 will have a useful life of at least 800 hours;
   (b) at least 15 will have a useful life of at least 800 hours;
   (c) at least 2 will not have a useful life of at least 800 hours.
5. Among the 15 cities that a professional society is considering for next 3 annual conventions, 5 are in the northern part of India. To avoid arguments, the selection is left to chance. If none of the cities can be chosen more than once, what are the probabilities that
   (a) none of the conventions will be held in the northern part,
   (b) all of the conventions will be held in the northern part?
6. A sortie of 20 aeroplanes is sent on operational flight. The chances that an aeroplane fails to return is 5%. Find the probability that (a) one plane does not return (b) at the most five planes do not return, (c) what is the most probable number of returns?
7. The following data shows the results of throwing 12 fair dice 4096 times; throw of 4, 5 or 6 being called success:

Success (x)	0	1	2	3	4	5	6	7	8	9	10	11	12
Frequency (f)	0	7	60	198	430	731	948	847	536	257	71	11	0

   Fit a binomial distribution and find the expected frequencies.
8. If a fair coin is tossed an even number of $2n$ times, show that the probability of obtaining more heads than tails is $\frac{1}{2}\left\{1 - C_n^{2n}\left(\frac{1}{2}\right)^{2n}\right\}$.
9. According to a genetic theory, a certain cross of guinea pigs will result in red, black, white and gray offspring in the ratio 4: 2: 3: 1. Find the probability that among 9 offspring 3 will be red, 2 black, 3 white and 1 gray.
10. As a student goes to school, he encounters a traffic signal which stays green for 35 seconds, yellow for 5 sec and red for 60 seconds. Assume that he goes to school each week day between 8:00 and 8:30 A.M. and X_1, X_2, X_3 be the number of times he encounters green, yellow and red signal, respectively. Find the joint distribution for (X_1, X_2, X_3).

11. In a test a light switch is turned on and off until it fails. If the probability that switch will fail any time it is turned on or off is 0.001, what is the probability that the switch will fail after it has been turned on or off 1,200 times? Assuming that the conditions for the geometric distribution are met.

12. The average number of accidents on a certain section of highway is two per week. Assuming it to follow Poisson distribution find the probability of (a) no accident on this section during a week period, (b) at most three accidents on this section during a two week period.

13. In a book of 520 pages, 390 typographical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

14. Suppose that in the production of radio resistors the probability of a resistor being defective is 0.1%. The resistors are sold in lots of 200, with the guarantee that all resistors are non-defective. What is probability that a given lot will violate this guarantee?

15. The probability that a person dies when he contracts a respiratory infection is 0.002. Of the next 2000 so infected, what is the mean number that will die? What is the S.D.?

16. The probability that a student pilot passes the written test for a pilot's licence is 0.7. Find the probability that the student will pass the test, (a) on the third try, (b) before the fourth try.

17. Twenty firms are under suspicion for violation of pollution norms but all cannot be inspected. Suppose that 3 of the firms are in violation. What is the probability that, (a) inspection of 5 firms find no violation. (b) will find two violations?

18. After correcting 50 pages of the proof of a book, the proof-reader finds that there are on an average 2 errors per 5 pages. How many pages would one expect to find with 0, 1, 2, 3 and 4 errors in 1000 pages of the first print of the book?

21.11 SPECIAL CONTINUOUS PROBABILITY DISTRIBUTIONS

In this section we consider some special continuous probability distributions like uniform, normal, exponential, gamma and beta, and study their various characteristics and applications.

21.11.1 Continuous Uniform Distribution

It is one of the simplest continuous distributions with constant (uniform) probability in a closed interval, say $[a, b]$.

If X is a continuous uniform r.v. defined on the interval $[a, b]$, then its p.d.f. is defined as

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases} \qquad ...(21.71)$$

It is clear that $\displaystyle\int_a^b f(x)dx = 1.$

We can easily see that

$$E(X) = \frac{a+b}{2}, \ \text{Var}(X) = \frac{(b-a)^2}{12}, \quad M_0(t) = \frac{e^{bt} - e^{at}}{t(b-a)}, \ t \neq 0. \qquad \text{...(21.72)}$$

Example 21.53: If X is uniformly distributed with mean 1 and variance 4/3, find $P(X < 0)$.

Solution: Let X is defined over $[a, b]$, then p.d.f is

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b,$$

Also, $$E(X) = \frac{a+b}{2} \text{ and } \text{Var}(X) = \frac{(b-a)^2}{12}.$$

Thus, $$\frac{a+b}{2} = 1 \text{ and } \frac{(b-a)^2}{12} = \frac{4}{3}.$$

Solving for a and b and using the fact that $a < b$, we get $a = -1$ and $b = 3$. Therefore,

$$f(x) = \frac{1}{4}; \quad -1 \le x \le 3.$$

Hence, $$P(X < 0) = \int_{-1}^{0} f(x)\,dx = \frac{1}{4}[x]_{-1}^{0} = 1/4.$$

Example 21.54: The metro trains on a certain section run every 10 minutes between 5 A.M. to 10 P.M. What is the probability that a commuter entering the station at a random time during this period will have to wait at least five minutes?

Solution: Let X be the waiting time in minutes, then X is distributed uniformly over [0, 10] with p.d.f.

$$f(x) = \begin{cases} \frac{1}{10}, & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

The probability that the waiting time will be at least five minutes is

$$P(X \ge 5) = \int_{5}^{10} \frac{1}{10}\,dx = \frac{1}{2}.$$

21.11.2 Normal Distribution

It is the most important continuous probability distribution in the field of statistics since in applications many random variables are normal random variables or they are approximately normal, particularly when the population size is large.

A continuous r.v. X with two parameters μ and σ having the p.d.f.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty \qquad ...(21.73)$$

is called the 'normal variate', and the distribution defined by (21.73) is called the 'normal distribution'. It defines a p.d.f., since

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz, \quad \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz, \text{ integrand being an even function in } z,$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t}\, t^{-\frac{1}{2}} dt, \quad \left(t = \frac{1}{2}z^2\right)$$

$$= \frac{1}{\sqrt{\pi}}\Gamma(1/2) = 1, \text{ since } \quad \Gamma(1/2) = \sqrt{\pi}.$$

The normal probability curve, $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is a bell-shaped curve symmetrical about the line $x = \mu$ and attains its maximum value of $1/\sigma\sqrt{2\pi} \approx 0.399/\sigma$ at $x = \mu$, refer Fig. 21.9.

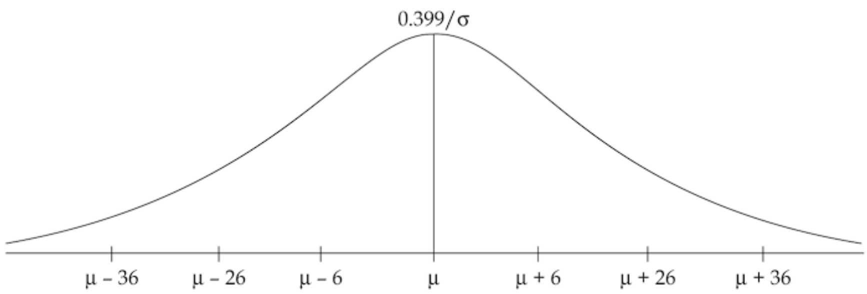


Fig. 21.9

$$f'(x) = -\frac{1}{\sigma^2}(x-\mu)f(x) \text{ and } f''(x) = \frac{-f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

Now $f'(x) = 0$ gives $x = \mu$ and at $x = \mu$, we have

$$f''(x) = -\frac{1}{\sigma^2}[f(x)]_{x=\mu} = -\frac{1}{\sigma^2} \cdot \frac{1}{\sqrt{2\pi}\sigma} < 0.$$

Hence, $x = \mu$ is the mode of the distribution.

The 'median' is that value of x which divides the distribution in two equal parts.

Thus for x to be median $\int_{-\infty}^{x} f(x)dx = \frac{1}{2}$. This gives $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$

or, $$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}. \qquad ...(21.76)$$

But, $$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz = \frac{1}{2}.$$

Hence, from (21.76), we have

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0,$$

which gives $x = \mu$. Hence for normal distribution

mean = mode = median.

Thus, the normal distribution is 'symmetrical'.

Moments about the mean: Odd order moments are given by

$$\mu_{2n+1} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{\sigma^{2n+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{1}{2}z^2} dz, \quad z = \frac{x-\mu}{\sigma}$$

$= 0$, integrand being an odd function of z.

Hence, $\mu_{2n+1} = 0$,

Thus in case of normal variate all odd order moments about the mean are zeros.

Even order moments are given by

$$\mu_{2n} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^{2n}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx = \frac{\sigma^{2n}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^{2n-1}e^{-\frac{1}{2}z^2}.zdz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}}\left[\left[-z^{2n-1}e^{-\frac{1}{2}z^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}(2n-1)z^{2n-2}e^{-\frac{1}{2}z^2}dz\right]$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}}(0-0) + (2n-1)\,\sigma^2.\frac{\sigma^{2n-2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^{2n-2}e^{-\frac{1}{2}z^2}dz$$

$$= (2n-1)\sigma^2\mu_{2n-2}.$$

Hence, $\mu_{2n} = (2n-1)\sigma^2\mu_{2n-2}$ ...(21.77)

It gives, $\mu_{2n} = (2n-1)(2n-3)\ \ldots\ 5.\,3.\,1.\,\sigma^{2n}$.

In particular, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$ etc.

Hence, $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3. \qquad \text{...(21.78)}$$

Thus the normal probability curve is 'symmetric and mesokurtic'.

Standard normal variate

If X is normally distributed with mean μ and variance σ^2, generally written as $X \sim N(\mu, \sigma^2)$ and if we define $Z = \frac{X-\mu}{\sigma}$, then

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}E(X-\mu) = 0 \quad \text{and} \quad \text{Var}\,(Z) = E(Z-\overline{Z})^2 = \frac{1}{\sigma^2}E(X-\overline{X})^2 = 1.$$

The variable Z defined so is called *standard normal variate* and its p.d.f. is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty. \qquad \text{...(21.79)}$$

Obviously the mean and variance of the standard variable are respectively zero and one, and it is denoted by $Z \sim N(0, 1)$.

The distribution function of a standard normal variate is given by

$$\Phi(z) = P\{Z < z\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{1}{2}z^2}dz. \qquad \text{...(21.80)}$$

The area property of normal probability integral

The probability of a normal variate lying between two values x_1 and x_2 is given by the area under the normal curve from x_1 to x_2, that is

$$P(x_1 \le X \le x_2) = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma}}dx = \frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{-\frac{1}{2}z^2}dz, \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{z_2} e^{-\frac{1}{2}z^2}dz - \int_0^{z_1} e^{-\frac{1}{2}z^2}dz\right]$$

$$= P(z_2) - P(z_1),$$

where the definite integral $P(z) = \frac{1}{\sqrt{2\pi}}\int_0^z e^{-\frac{1}{2}z^2}dz$ is known as *normal probability integral* and gives the *area under standard normal curve between the ordinates at Z = 0 and Z = z*. These areas have been tabulated for different values of z at intervals of 0.01 and are given at Table *I* (p. 527).

In particular, $P(\mu - \sigma < X < \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)dx$ can be evaluated as

$$P(-1 < Z < 1) = \int_{-1}^{1} \phi(z)dz = \frac{2}{\sqrt{2\pi}}\int_0^1 e^{-\frac{1}{2}z^2}dz = 2 \times 0.3413 = 0.6826, \text{ from Table } I$$

Similarly,

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2) = \frac{2}{\sqrt{2\pi}}\int_0^2 e^{-\frac{1}{2}z^2}dz = 2(0.4772) = 0.9544$$

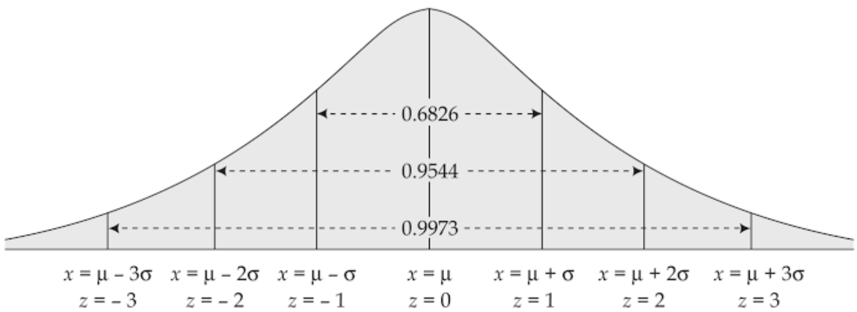


Fig. 21.10

and $P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3) = \dfrac{2}{\sqrt{2\pi}} \int_0^3 e^{-\frac{1}{2}z^2} dz = 2(0.49865) = 0.9973.$

Hence, the probability that a normal variate X lies outside the region $\mu \pm 3\sigma$ is given by

$$P(|x - \mu| > 3\sigma) = P(|z| > 3) = 1 - P(-3 \le z \le 3) = 1 - 0.9973 = 0.0027.$$

Thus though theoretically normal variate ranges from $-\infty$ to ∞, yet in all probability we should expect it to lie within the range $\mu \pm 3\sigma$, as shown in Fig. 21.10.

Remarks:

1. Since in table we are given the areas under standard normal curve thus in numerical problems we convert the variable in its standard form.
2. From the symmetry of the normal probability curve we have $P(Z > z) = P(Z < -z)$, as shown in Fig. 21.11.

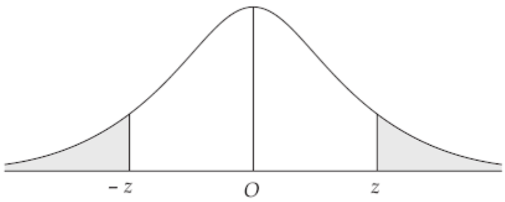


Fig. 21.11

3. From Table *I* we observe that $P(-1.96 < Z < 1.96) = 0.95$ and $P(-2.58 < Z < 2.58) = 0.99$, and these are two important values to remember.

Example 21.55: If the amount of cosmic radiations to which a person exposed while flying across a specific continent is a normal random variable with mean 4.35 units and S.D. 0.59 units. Find the probabilities that the amount of exposure during such a flight is

(a) between 4.00 and 5.00 units, (b) at least 5.50 units.

Solution: Let X be the amount of cosmic radiations exposed, we define

$$Z = \frac{X - 4.35}{0.59} \sim N(0, 1).$$

We have,
$$\begin{aligned} P(4 < X < 5) &= P(-0.59 < Z < 1.10) \\ &= P(1.10) + P(0.59) \\ &= 0.3643 + 0.2224 = 0.5867, \text{ from Table } I. \end{aligned}$$

The area is as shown in Fig. (21.12a).

(b)
$$\begin{aligned} P(X \ge 5.50) &= P(Z \ge 1.95) \\ &= 0.5 - P(1.95) \\ &= 0.5 - 0.4744 = .0256, \text{ from Table } I. \end{aligned}$$

The area is as shown in Fig. (21.12b).

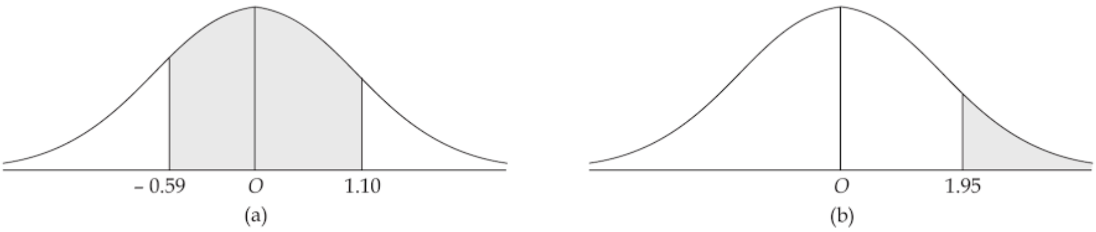


Fig. 21.12

Example 21.56: In a production of iron rods the diameter X can be approximated to be normally distributed with mean 2 inches and S.D. 0.008 inches.

(a) What percentage of defectives can we expect if we set the acceptance limits at 2 ± 0.02 inches?

(b) How should we set the acceptance limits to allow for 4% defectives?

Solution: (a) Define $Z = \dfrac{X-2}{0.008} \sim N(0, 1)$.

Then $P(1.98 \le X \le 2.02) = P(-2.5 \le Z \le 2.5) = 2P(0 \le Z \le 2.5) = 2P(2.5)$

From Table I, $P(2.5) = 0.4938$, hence $P(1.98 \le x \le 2.02) = 0.9876$

This gives $P(|X-2| > 0.02) = 1 - 0.9876 = .0124$.

Hence the percentage of defectives expected is 1.24%.

(b) Let the acceptance limits be fixed at $2 \pm k$, then

$$P(2 - k \le X \le 2 + k) = 0.96$$

or, $$P\left(\frac{-k}{0.008} < Z < \frac{k}{0.008}\right) = 0.96, \quad \text{or} \quad P\left(\frac{k}{0.008}\right) = 0.48,$$

which gives $\dfrac{k}{0.008} = 2.054$, or $k = .016432$.

The acceptance limits should be set at 2 ± 0.0164, that is, the interval [1.9836, 2.0164].

Example 21.57: A company has installed 10,000 electric lamps in a metro. If these lamps have an average life of 1,000 burning hours with a S.D. of 200 hours. Assuming normality, what number of lamps might be expected to fail

(a) in the first 800 burning hours.

(b) between 800 and 1200 burning hours.

After what period of burning hours would you expect that

(c) 10% of the lamps would fail?

(d) 10% of the lamps would survive?

Solution: Let X be the life of a bulb in burning hours. Define $Z = \dfrac{X - 1000}{200}$, then $Z \sim N(0, 1)$.

(a) $$P(X < 800) = P(Z < -1) = P(Z > 1) = 0.5 - P(0 < Z < 1)$$
$$= 0.5 - 0.3413 = 0.1587, \text{ form Table } I.$$

Therefore, out of 10,000 bulbs it is expected that 1587 will fail in the first 800 hours.

(b) $P(800 < X < 1200) = P(-1 < Z < 1) = 2P(0 < Z < 1) = 0.6826.$

Therefore, out of 10,000 bulbs it is expected that 6826 will burn between 800 and 1200 hours.

(c) If 10% of the bulbs fail after x_1 hours of burning, then x_1 be such that $P(X < x_1) = 0.10$. When $x = x_1$, then $z = \dfrac{x_1 - 1000}{200} = -z_1$, say. We find z_1 such that

$$P(Z < -z_1) = 0.10, \text{ or } P(Z > z_1) = 0.10, \text{ or } P(0 < Z < z_1) = 0.40.$$

From Table I, $z_1 = 1.28$. Thus, $\dfrac{x_1 - 1000}{200} = -1.28.$

It gives $x_1 = 1000 - 256 = 744.$

Therefore after 744 hours of burning life, 10% of the bulbs are likely to fail.

(d) If 10% of the bulbs are still burning after x_2 hours of burning, then x_2 be such that $P(X > x_2) = 0.10$. When $x = x_2$, then $z = (x_2 - 100)/200 = z_2$, say.

It gives $P(Z > z_2) = 0.10$ or, $P(0 < Z < z_2) = 0.40.$

From Table I $z_2 = 1.28$. Hence, $\dfrac{x_2 - 100}{200} = 1.28$, which gives, $x_2 = 1256.$

Thus, 10% of the bulbs are likely to burn after 1256 hours of the burning life.

Fitting of normal distribution.

To fit normal distribution to the given data we first calculate the mean μ and S.D. σ from the given data. Then the normal probability curve to be fitted to the given data is given by

$$y = f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

Example 21.58: Obtain the equation of the normal curve that may be fitted to the following data. Also obtain the expected normal frequencies.

Class :	60–65	65–70	70–75	75–80	80–85	85–90	90–95	95–100
Frequency :	3	21	150	335	326	135	26	4

Solution: First we calculate the mean and S.D. for the given data

Class	*Frequency*	*Mid pt. (x)*	$u = \dfrac{x-77.5}{5}$	*fu*	*fu*^{2}
60 – 65	3	62.5	–3	–9	27
65 – 70	21	67.5	–2	–42	84
70 – 75	150	72.5	–1	–150	150
75 – 80	335	77.5	0	0	0
80 – 85	326	82.5	1	326	326
85 – 90	135	87.5	2	270	540
90 – 95	26	92.5	3	78	234
95 – 100	4	97.5	4	16	64
	1000			489	1425

$$\bar{u} = \frac{489}{1000} = 0.489, \qquad \sigma_u^2 = \frac{1425}{1000} - (0.489)^2 = 1.425 - 0.239 = 1.186$$

Thus, $\bar{x} = a + h\bar{u} = 77.5 + 5(0.489) = 77.5 + 2.445 = 79.945.$

and, $\sigma_x = h\sigma_u = 5(1.089) = 5.445.$

Hence, the equation of the normal curve to be fitted to the given data is given by

$$f(x) = \frac{1}{5.445\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-79.945}{5.445}\right)^2}. \qquad \text{...(21.81)}$$

To calculate the theoretical frequencies we calculate the area under the probability curve (21.81) in the interval (x_1, x_2), or (z_1, z_2) as given by

$$\Delta z = \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-z^2/2} dz - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz,$$

where $z = \dfrac{x - 79.945}{5.445}$.

By using the valuse from Table *I*, we form the following table:

(x_1, x_2)	*(z_1, z_2)*	*Area $\Delta z = P(z_2) - P(z_1)$*	*Expected Frequency = $N\Delta z$*
(–∞, 60)	(–∞, –3.663)	0.00011	0.11 ≈ 0
(60, 65)	(–3.663, –2.745)	0.00291	2.91 ≈ 3
(65, 70)	(–2.745, –1.826)	0.03104	31.04 ≈ 31
(70, 75)	(–1.826, –0.908)	0.14787	147.87 ≈ 142
(75, 80)	(–0.908, –0.010)	0.32205	322.05 ≈ 322
(80, 85)	(–0.010, 0.928)	0.31930	319.30 ≈ 319
(85, 90)	(0.928, 1.487)	0.14407	144.07 ≈ 144
(90, 95)	(1.487, 2.675)	0.02979	29.79 ≈ 30
(95, 100)	(2.675, 3.683)	0.00273	2.73 ≈ 3
(100, ∞)	(3.683, ∞)	0.00011	0.11 ≈ 0
Total			1000

The normal approximation as a limiting case of binomial

If in case of binomial distribution n is large and neither p nor q is small enough to use the Poisson approximation, then distribution can be approximated to normal. In fact, we have the following result which we state without proof.

If x is a binomial variate with parameters n and p, then the limiting form of the p.d.f. of the standardized r.v. $z = \dfrac{x - np}{\sqrt{np(1-p)}}$ *as* $n \to \infty$ *is the standard normal distribution given by* $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty < z < \infty.$

We should note that although x takes on only the values 0, 1, 2, ..., n, in the limiting case as $n \to \infty$, the standardized variable z is continuous and takes value from $-\infty$ to ∞, with probability density as standard normal density.

Example 21.59: A 20% of the memory chips made in a certain plant are defective. What are the probabilities that in a lot of 100 randomly chosen for inspection

(a) at most 15 will be defective?

(b) exactly 15 will be defective?

Solution: Here mean $\mu = 100(0.20) = 20$ and S.D. $\sigma = \sqrt{100(0.20)\,(0.80)} = 4$, thus the binomial variable may be approximated by $X \sim N(20, 16)$, a normal variate with mean 20 and variance 16.

Since the variable under consideration is discrete to 'spread' its values over a continuous scale we represent each value k by the interval (k – ½, k + ½). Thus 15 is represented as 14.5 – 15.5.

(a) $$P(X < 15.5) = P(Z < -1.13) = P(Z > 1.13)$$
$$= 0.5 - P(0 < Z < 1.13) = 0.5 - 0.3708 = 0.1292, \text{using Table } I$$

(b) $$P(14.5 < X < 15.5) = P(-1.38 < Z < -1.13) = P(1.13 < Z < 1.38)$$
$$= 0.4162 - 0.3708 = .0454, \text{ using Table } I.$$

Remark: In case calculations are worked out using binomial distribution, for part (a) we arrive at 0.1285 and for part (b) 0.0481, both in close with the respective approximations obtained. In general, the normal approximation to the binomial distribution is advisable when both np and $n(1 - p)$ are greater than 15.

Example 21.60: A multiple-choice quiz has 200 questions each with 4 possible answers of which only 1 is the correct answer. What is the probability that sheer guess-work yields from 25 to 30 correct answer for 80 of the 200 problems about which the student has no knowledge.

Solution: We have, $n = 80$, $p = \dfrac{1}{4}$, thus

$$\mu = np = 20 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{80 \times \frac{1}{4} \times \frac{3}{4}} = 3.873.$$

Let X be the normal approximation to the underlying binomial variate, then $X \sim N(20, 15)$, a normal variate with mean 20 and variance 15.

To spread the variable over the continuous scale we find the probability that X lies in the interval 24.5 – 30.5. We define $z = \dfrac{x-20}{3.873}$. Thus,

$$P(24.5 < x < 30.5) = P(1.16 < z < 2.71) = 0.4966 - 0.3770 = 0.1196.$$

EXERCISE 21.5

1. A conference room can be reserved for no more than five hours. Assuming that duration X of a conference has a uniform distribution over the interval [0, 5]. What is the p.d.f.? What is the probability that any given conference lasts at most 4 hours?

2. The daily amount of coffee in litres dispensed by a machine at a plaza is uniformly distributed between 7 litres to 10 litres. Find the probability that on a given day the amount of coffee dispensed by this machine will be

(a) at most 8.8 litres,

(b) more than 7.4 litres but less than 9.5 litres,

(c) at least 8.5 litres.

3. If X is normally distribution with mean 18 and S.D. 2.5, find

(a) $P(X < 15)$ (b) $P(17 < X < 21)$

(c) the value of k such that $P(X < k) = 0.2236$,

(d) the value of k such that $P(X > k) = 0.1814$.

4. The actual amount of instant coffee that a filling machine put into '4-ounce' jar may be approximated as a normal random variable with S.D. 0.04 ounces. If only 2% of the jars are to contain less than 4 ounces what should be the mean fill of these jars?

5. If in a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution.

6. In a test on 200 electric bulbs, it was found that the life of a particular make was normally distributed with mean 2040 hours and S.D. 60 hours. Estimate the number of bulbs likely to burn for

(a) more than 2150 hours,

(b) less than 1950 hours, and

(c) more than 1920 hours but less than 2160 hours.

7. The average life of an inverter is 10 year with a S.D. of 2 year. The manufacturer replaces free all inverters that fail while under guarantee. If he is willing to replace only 3% of the inverters that fail, how long a guarantee should he offer, assuming that the lifetime follows a normal distribution.

8. If the lifetime of a certain kind of automobile battery is normally distributed with a mean of 5 years and a S.D. of 1 year, and the manufacturer wishes to guarantee the battery for 4 years, what percentage of the batteries will he have to replace under the guarantee?

9. If the mathematics score of an entrance exam are normally distributed with mean 480 and S.D. 100 and if an institution sets 500 as the minimum score for new students, what per cent of students would not reach that score?

10. Cerebral blood flow (CBF) in the brains of healthy people is normally distributed with a mean of 74 and a S.D. of 16.

(a) What percentage of healthy people will have CBF readings between 60 and 80?

(b) If a person has a CBF reading below 40 he is classified at risk for a stroke. What proportion of healthy people will mistakenly be diagnosed as "at risk"?

11. Suppose that the amount of money spent by shoppers at a mall between 4 P.M. to 6 P.M. on Sunday is normally distributed with mean of Rs. 4250 and a S.D. of Rs. 500. A shopper is randomly selected on a Sunday between 4-6 P.M. and asked about his spending pattern

(a) What is the probability that he has spent more than Rs. 4500 at the mall?

(c) What is the probability that he has spent between Rs. 4500 and Rs. 5000 at the mall?

(d) If two shoppers are randomly selected, what is the probability that both have spent more than Rs. 5000 at the mall?

12. Fit a normal distribution to the following data and calculate the expected frequencies

Class	1-3	3-5	5-7	7-9	9-11
Frequency	1	4	6	4	1.

13. The following table gives baseball throw for a distance by 303 first year students of a college

(a) Fit a normal distribution and find the theoretical frequencies.

(b) Find the expected number of students throwing baseballs at a distance exceeding 105 feet on the basis that the data fits a normal distribution.

Distance in feet	*Number of students*
15 – 25	1
25 – 35	2
35 – 45	7
45 – 55	25
55 – 65	33
65 – 75	53
75 – 85	64
85 – 95	44
95 – 105	31
105 – 115	27
115 – 125	11
125 – 135	4
135 – 145	1

14. A sample of 100 items is taken from a batch known to contain 40% defectives. Using normal approximations find the probability that sample contains:

(a) at least 44 defectives, (b) exactly 44 defectives.

15. A certain drug is effective in 72% of cases. Given 2,000 patients are treated with drug, what is the probability that it will be effective for, (a) at least 1,400 patients, (b) less than 1,300 patients, (c) exactly 1,420 patients.

16. A process for manufacturing an electronic component is 1% defective. A quality control plan is to select 100 items from the process, and if none is defective the process continues. Using normal approximation, find the probability that the process continues

(a) for the sample plan described

(b) even if the process has gone bad to produce 5% defective.

21.11.3 Exponential Distribution

A continuous random variable X with probability density function $f(x)$ defined by

$$f(x) = \begin{cases} ae^{-ax}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0, \end{cases} \qquad ...(21.82)$$

for some $a > 0$, is called an *exponential variate with parameter 'a'* and the distribution is said to be *exponential distribution*. The function (21.82) defines a probability density function, since

$$\int_{-\infty}^{\infty} f(x)dx = a\int_{0}^{\infty} e^{-ax}dx = [-e^{-ax}]_0^{\infty} = 1.$$

The distribution function $F(x)$ of an exponential variate is given by

$$F(x) = P(X \leq x)$$

$$= \int_0^x ae^{-ax}dx = (1 - e^{-ax}), \quad x \geq 0. \qquad ...(21.83)$$

Also we can calculate that

$$E(X) = 1/a, \qquad \text{Var }(X) = 1/a^2 \text{ and } \qquad M_0(t) = \frac{a}{a-t}, \quad t < a \qquad ...(21.84)$$

Thus, the mean is reciprocal of the parameter 'a' and the variance is equal to the square of the mean in case of an exponential variate.

The exponential distribution arises as the distribution of amount of time until some specific event occurs. For example, the amount of time starting from now a car comes for service at a service station, a new war breaks out, a patient comes at an emergency reception, etc. are all random variables that behave exponentially.

An important property of the exponential distribution is that it "*lacks memory*", that is, if X has an exponential distribution, then

$$P(X > s + t \mid X > t) = P(X > s) \qquad ...(21.85)$$

for all $s, t > 0$.

Since (21.85) can be written as

$$\frac{P(X > s + t \text{ and } X > t)}{P(X > t)} = P(x > s)$$

or,
$$P(X > s + t) = P(X > s)P(X > t),$$

which is satisfied when X has exponential distribution (21.82).

In case we interpret x as the lifetime of some equipment in hours, then (21.85) simply means that the probability that the equipment survives for at least $(s + t)$ hours given that it has survived t hours is the same as the initial probability that it survives for at least s hours.

Example 21.61: A system contains a certain type of component whose life-time X is exponentially distributed with mean of 5 years. If 8 such components are installed in different systems, then what is the probability that at least 3 are still working at the end of 7 years?

Solution: The p.d.f for r.v. X is given by

$$f(x) = \frac{1}{5} e^{-x/5}, \quad x \geq 0.$$

Thus,
$$P(X > 7) = \frac{1}{5}\int_7^\infty e^{-x/5}dx = e^{-7/5} = 0.1827.$$

If n represents the number of components out of 8 working after 7 years of instalment, then

$$P(n \geq 3) = \sum_{n=3}^{8} C_n^8 (0.1827)^n (0.8173)^{8-n}$$
$$= 1 - [C_0^8 (0.8173)^8 + C_1^8 (0.1827)(0.8173)^7 + C_2^8 (0.1827)^2 (0.8173)^6]$$
$$= 1 - [0.1991 + 0.3560 + 0.2786] = 0.1663.$$

Example 21.62: If on the average three trucks arrive per hour to be unloaded at a warehouse, using exponential distribution find the probabilities that the time between the arrival of successive trucks will be, (a) less than 5 minutes, (b) at least 45 minutes.

Solution: Let the r.v.t denote the time in hrs between arrival of successive trucks then its p.d.f is

$$f(t) = 3e^{-3t}, \ 0 \leq t < \infty$$

(a)
$$P(0 < t < 1/12) = \int_0^{1/12} 3e^{-3t}dt = 1 - e^{-1/4} = 0.221.$$

(b)
$$P\left(\frac{3}{4} < t < \infty\right) = \int_{3/4}^{\infty} 3e^{-3t}dt = e^{-9/4} = 0.105.$$

21.11.4 The Gamma Distribution

A continuous random variable X with probability density function, for some $\alpha > 0$, $\beta > 0$, defined by

$$f(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{...(21.86)}$$

is called *gamma variate with parameters* (α, β) and the distribution is called *gamma distribution*. Here $\Gamma(\alpha)$ is the value of the gamma function with parameter $\alpha > 0$, given by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

We have $\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1)$ and when α is positive integer, then $\Gamma(\alpha) = (\alpha - 1)!$

The function (21.86) defines a p.d.f., since

$$\int_{-\infty}^\infty f(x)dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-\beta x}dx = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}dt = 1, \quad t = \beta x.$$

We can show that

$$E(X) = \alpha/\beta, \quad \text{Var}\,(X) = \alpha/\beta^2 \;\text{ and }\; M_0(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha. \qquad ...(21.87)$$

We note that exponential p.d.f defined by (21.82) is a special case of gamma p.d.f (21.86) for $\alpha = 1$.

The relationship between the gamma and the exponential distribution allows the gamma function to find applications similar to that of exponential in particular in the field of queuing theory and reliability problems. In addition to this, gamma distribution is frequently used in 'life-testing', the waiting time until "death" probability models.

Remark: Sometimes
$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} x^{\alpha-1}e^{-x}, & x \geq 0,\ \alpha > 0 \\ 0, & \text{otherwise} \end{cases} \qquad ...(21.88)$$

is also defined as the p.d.f of the gamma variate x with parameter α.

Example 21.63: The daily consumption of milk in a city, in excess of 20,000 litres, is approximately distributed as a gamma variate with parameters $\alpha = 2$ and $\beta = 1/10{,}000$. The city has a daily stock of 30,000 litres. What is the probability that the stock is insufficient on a particular day?

Solution: If X denotes the daily consumption in excess of 20,000 litres, then p.d.f of X is

$$f(x) = \frac{1}{(10{,}000)^2\,\Gamma(2)}\, x^{2-1} e^{-x/10{,}000}, \; x > 0.$$

The stock of 30,000 litres will be insufficient on a particular day, if the excess consumption is more than 10,000 litres.

Thus,
$$P(X > 10{,}000) = \int_{10{,}000}^\infty \frac{xe^{-x/10{,}000}}{(10{,}000)^2}dx = \int_1^\infty te^{-t}dt, \quad t = x/10{,}000$$
$$= 2/e = 0.736$$

21.11.5 The Beta Distribution

A continuous random variable X with probability density function, for some $\alpha > 0, \beta > 0$, defined by

$$f(x) = \begin{cases} \dfrac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \qquad ...(21.89)$$

is called *beta variable with parameters α and β* and the distribution is called *beta distribution*, sometimes *beta distribution of the first kind*. Here $B(\alpha, \beta)$ is the value of the beta function given by

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx.$$

The relation between beta and gamma function is $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$. Thus, for positive integral values of α and β, we have $B(\alpha, \beta) = (\alpha-1)!\,(\beta-1)!/(\alpha+\beta-1)!$.

Obviously the function defined by (21.89) is p.d.f., since

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{B(\alpha, \beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{B(\alpha, \beta)}{B(\alpha, \beta)} = 1.$$

Using the properties of beta and gamma functions we can show that

$$E(X) = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \text{Var}\,(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \qquad ...(21.90)$$

Example 21.64: If the proportion of a brand of television set requiring service during the first year of operation is a random variable having a beta distribution with $\alpha = 3$ and $\beta = 2$, what is the probability that at least 80% of the new models sold this year of this brand will require service during the first year of operation?

Solution: If the r.v. X denotes the proportion of T.V. set requiring service during the first year of operation, then its p.d.f. is

$$f(x) = \frac{1}{B(3, 2)} x^2(1-x), \; 0 < x < 1.$$

Thus, $$P(x > 0.8) = \frac{1}{B(3, 2)} \int_{0.8}^{1} x^2(1-x)dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{0.8}^{1} = 0.1808$$

EXERCISE 21.6

1. The amount of time that a surveillance camera will run without having to be reset is random variable having exponential distribution with an average of 60 days. Find the probability that such a camera will have to be reset, (a) in less than 60 days, (b) at least 50 days.

2. The length of time for one individual to be served at a canteen is a random variable having an exponential distribution with mean of 4 minutes. What is the probability that a person is served in less than 3 minutes on at least 4 of the next 6 visits?

3. A continuous r.v X has the p.d.f. $f(x) = \begin{cases} Ae^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$

   Find the value of A and show that for any two positive numbers s and t,
   $$P[X > s + t \mid X > s] = P[X < t].$$

4. Following data gives the burning hours of 200 bulbs. Calculate the theoretical frequencies on the basis that burning hours of a bulb is exponentially distributed random variable.

Burning hrs	0-20	20-40	40-60	60-80	80-100
Number of bulbs	104	56	24	12	4

5. For a certain dose of the toxicant, a study on mice determines that the survival time, in weeks, has a gamma distribution with parameter $\alpha = 5$. What is the probability that a mouse survives no longer than 60 weeks?

6. The survival time in weeks of an animal when subjected to certain exposure of gamma radiation has a gamma distribution with $\alpha = 5$ and $\beta = 1/10$.
   (a) What is the mean survival time of a randomly selected animal of the type used in the experiment?
   (b) What is the probability that an animal survives more than 30 weeks?

7. Suppose that proportion of defectives, supplied by a vendor from lot to lot may be looked upon as a random variable having the beta distribution with $\alpha = 2$ and $\beta = 3$.
   (a) Find the average proportion of defectives in a lot from this vendor.
   (c) Find the probability that a lot from this vendor will contain 30% or more defectives.

8. The response time of a certain computer system in seconds has an exponential distribution with a mean of 3 seconds.
   (a) What is the probability that response time exceeds 5 seconds?
   (b) What is the probability that response time is between 5-10 seconds?

21.12 METHOD OF LEAST SQUARES AND CURVE FITTING

Fitting of curve to a given bivariate data is important both from the point of view of theoretical and practical statistics. Theoretically, concept is useful in the study of correlation and regression and practically, functional relationship between x and y enables us to predict the response y for a specific input x. The appropriate relationship to be fitted may be polynomial, algebraic, exponential or logarithmic depending upon the nature of the data. Method of least square is an excellent technique for fitting an appropriate relationship to the given data.

21.12.1 Method of Least Squares

Suppose we have n paired observations (x_i, y_i), $i = 1, 2, ..., n$, say, marks in mathematics and physics for a group of n individuals in the end-semester examination. Let the nature of the problem under consideration suggests a linear relation between x and y. We want to determine the line which, in

some sense, provides the '*best-fit*'. Such a line may be useful for estimating values for y for some specific values of x. This line of '*best-fit*' is obtained by applying the *method of least squares* which in this case may be stated as follows.

"*The straight line*

$$y = a + bx \qquad ...(21.91)$$

should be fitted through the given points (x_i, y_i), $i = 1, 2, \ldots, n$ *so that sum of the squares of deviations (errors) of these points from the straight line is minimum, where the deviations are measured in the vertical direction*".

Referring to Fig. 21.13, the point M_i on the straight line with abscissa x_i has the ordinate $a + bx_i$. Hence, the deviation from P_i is $M_iP_i = (y_i - a - bx_i)$ and the sum of the squares of these deviations is

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2 \qquad ...(21.92)$$

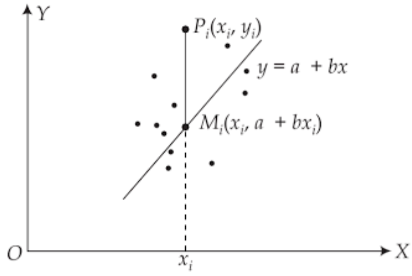


Fig. 21.13

We are to determine a and b such that S is minimum.

Differentiating (21.92) w.r.t a and b, respectively we obtain

$$\frac{\partial S}{\partial a} = -2\sum_{i=1}^{n} (y_i - a - bx_i)$$

$$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n} x_i(y_i - a - bx_i).$$

Equating these separately to zero and rearranging the terms we obtain

$$\left.\begin{aligned} \sum_{i=1}^{n} y_i &= na + b\sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i &= a\sum x_i + b\sum_{i=1}^{n} x_i^2 \end{aligned}\right\} \qquad ...(21.93)$$

These equations are called the *normal equations* for fitting the straight line (21.91). Solving (21.93) for a and b and substituting the values obtained, the line (21.91) is the line of '*best-fit*' in the sense of least square.

21.12.2 Fitting of Polynomial of Degree k

The method discussed above can be generalized to fit a polynomial of degree k, ($k \le n - 1$), given by

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k,\ a_k \neq 0 \qquad ...(21.94)$$

Example 21.66: Fit a parabola of second degree to the following data:

x :	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y :	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Solution: Let the parabola be $y = a + bu + cu^2$, where $u = \dfrac{x - 2.5}{0.5}$.

The normal equations are

$$\Sigma y = 7a + b\Sigma u + c\Sigma u^2$$
$$\Sigma uy = a\Sigma u + b\Sigma u^2 + c\Sigma u^3$$
$$\Sigma u^2 y = a\Sigma u^2 + b\Sigma u^3 + c\Sigma u^4$$

We formulate the following table

x	$u = \dfrac{x - 2.5}{0.5}$	y	u^2	u^3	u^4	uy	u^2y
1.0	–3	1.1	9	–27	81	–3.3	9.9
1.5	–2	1.3	4	–8	16	–2.6	5.2
2.0	–1	1.6	1	–1	1	–1.6	1.6
2.5	0	2.0	0	0	0	0	0
3.0	1	2.7	1	1	1	2.7	2.7
3.5	2	3.4	4	8	16	6.8	13.6
4.0	3	4.1	9	27	81	12.3	36.9
Total	0	16.2	28	0	196	14.3	69.9

Substituting for Σu, Σu^2, Σu^3, Σu^4, Σy, Σuy and $\Sigma u^2 y$ in the normal equations, we obtain

$$7a + 28c = 16.2, \quad 28b = 14.3, \quad \text{and} \quad 28a + 196c = 69.9.$$

Solving for a, b and c, we obtain $a = 2.07$, $b = 0.511$ and $c = 0.061$.

Hence the parabolic curve of 'best-fit' to the given data is

$$y = 2.07 + 0.511\left(\frac{x - 2.5}{0.5}\right) + 0.061\left(\frac{x - 2.5}{0.5}\right)^2,$$

or, $$y = 1.04 - 0.2x + 0.24x^2.$$

21.12.3 Fitting of Non-polynomial Curves

Using the concept of fitting of straight line, we can fit the curves of the form

$$y = ax^b, \quad y = ae^{bx}, \quad \text{or} \quad xy^a = b.$$

For example, to fit the curve $y = ax^b$ take logarithm both sides, we obtain

$$\ln y = \ln a + b \ln x.$$

or, $$Y = A + bX,$$

where $Y = \ln y$, $A = \ln a$, and $X = \ln x$.

We obtain the normal equations for this straight line and solve those equations for A and b. From A we calculate for a. Substituting the values for a and b we obtain the desired curve. Similarly, we can proceed for other curves.

Example 21.67: Determine the constants a and b such that $y = ae^{bx}$ is the 'best-fit' curve for the following data.

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

Solution: The curve is $y = ae^{bx}$. Taking logarithm both sides, we obtain

$$\ln y = \ln a + bx,$$

or,
$$Y = A + bx,$$

where
$$Y = \ln y \text{ and } A = \ln a$$

We formulate the following table:

	x	$Y = \ln y$	x^2	xY
	2	1.405	4	2.810
	4	2.405	16	9.620
	6	3.405	36	20.430
	8	4.405	64	35.240
	10	5.405	100	54.050
Total	30	17.025	220	122.150

Substituting for Σx, ΣY, Σx^2 and ΣxY in the normal equations for the straight line $Y = A + bx$, we obtain

$$5A + 30b = 17.025.$$

$$17.025A + 220b = 122.50$$

Solving for A and b, we obtain $A = 0.405$ and $b = 0.5$. Hence, $a = e^{0.405} = 1.499$.

Thus the curve is $y = 1.499x^{0.5}$.

EXERCISE 21.7

1. Fit a straight line of the form $y = a + bx$ to the data

x :	1	2	3	4	6	8
y :	2.4	3.1	3.5	4.2	5.0	6.0

2. If the straight line $y = a + bx$ is the best-fit line to the set of points (x_i, y_i), $i = 1, 2, \ldots, n$, then show that

$$\begin{vmatrix} x & y & 1 \\ \Sigma x_i & \Sigma y_i & n \\ \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \end{vmatrix} = 0.$$

21.13.1 Correlation

Correlation is a measure of the degree of association existing between two variables in a bivariat data. The simplest way to get an idea whether the variables are correlated is to obtain the diagram of the points (x_i, y_i), $i = 1, 2, ..., n$ called the *scatter diagram*, refer Fig. 21.15.

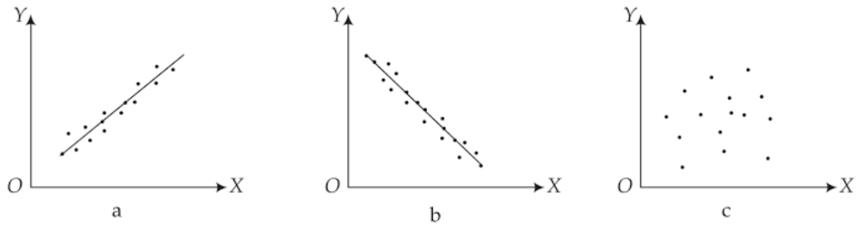


Fig. 21.15

If all the points on the scatter diagram lie on a straight line, then *perfect linear correlation* is said to exist. When the points tend to concentrate about a straight line with positive gradient, then *positive linear correlation* is said to exist, refer Fig. 21.15a and when the points tend to concentrate about a straight line with negative gradient, then *negative linear correlation* is said to exist, refer Fig. 21.15b. In case the points are widely scattered, as shown in Fig. 21.15c, then a *poor correlation* or *no correlation* is expected. In positive linear correlation, increase (decrease) in x is accompanied with the increase (decrease) in y and vice versa; while in negative linear correlation increase (decrease) in x is accompanied with decrease (increase) in y and vice versa.

Karl Pearson's coefficient of linear correlation

The Karl Pearson's correlation coefficient between two random variables X and Y, denoted by r_{xy}, is a numerical measure of *linear relationship* between them and is defined as

$$r_{xy} = \frac{\text{Cov}\,(x, y)}{\sigma_x\,\sigma_y} = \frac{E(x-\overline{x})\,(y-\overline{y})}{\sigma_x\,\sigma_y} \qquad ...(21.95)$$

If (x_i, y_i), $i = 1, 2, ..., n$ is the bivariate distribution, then

$$\text{Cov}\,(x, y) = \frac{1}{n}\Sigma\,(x_i-\overline{x})\,(y_i-\overline{y}) = \frac{1}{n}\Sigma\,x_iy_i - \left(\frac{1}{n}\Sigma\,x_i\right)\left(\frac{1}{n}\Sigma\,y_i\right),$$

$$\sigma_x^2 = \frac{1}{n}\Sigma\,(x_i-\overline{x})^2 = \frac{1}{n}\Sigma\,x_i^2 - \left(\frac{1}{n}\Sigma x_i\right)^2,$$

$$\sigma_y^2 = \frac{1}{n}\Sigma\,(y_i-\overline{y})^2 = \frac{1}{n}\Sigma\,y_i^2 - \left(\frac{1}{n}\Sigma\,y_i\right)^2.$$

Thus a convenient expression for the Karl-Pearson's coefficient of correlation for calculation purpose is given by

$$r_{xy} = \frac{\Sigma (x - \bar{x})(y - \bar{y})}{\sqrt{\Sigma (x - \bar{x})^2} \cdot \sqrt{\Sigma (y - \bar{y})^2}}$$

$$= \frac{\Sigma xy - \dfrac{(\Sigma x)(\Sigma y)}{n}}{\sqrt{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}} \cdot \sqrt{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}}}. \quad ...(21.96)$$

Remark. The coefficient r_{xy} provides a measure of only the linear relationship between X and Y. It is not suitable for non-linear relationship.

Limits for correlation coefficient

We have, $$E\left[\left(\frac{x - \bar{x}}{\sigma_x}\right) \pm \left(\frac{y - \bar{y}}{\sigma_y}\right)\right]^2 \geq 0.$$

This gives $$E\left(\frac{x - \bar{x}}{\sigma_x}\right)^2 + E\left(\frac{y - \bar{y}}{\sigma_y}\right)^2 \pm 2\frac{E(x - \bar{x})(y - \bar{y})}{\sigma_x \sigma_y} \geq 0$$

or, $1 + 1 \pm 2 r_{xy} \geq 0$, or $-1 \leq r_{xy} \leq 1$.

Thus, r_{xy} lies between –1 and + 1. If r = +1, the correlation is perfect and positive, and if r = –1, the correlation is perfect and negative. If r = 0, then there is no linear correlation between x and y and variables are said to be uncorrelated.

Effect of change of origin and scale

Let $u = \dfrac{x - a}{h}$ and $v = \dfrac{y - b}{k}$, where a, b, h, k are constants with $h, k > 0$.

We have $x = a + hu$, $y = b + kv$. We can very easily show that

$$\text{Cov}(x, y) = hk \, \text{Cov}(u, v)$$

Also, $\sigma_x = h\sigma_u$ and $\sigma_y = k\sigma_v$. Hence,

$$r_{xy} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = \frac{\text{Cov}(u, v)}{\sigma_u \sigma_v} = r_{uv} \quad ...(21.97)$$

Thus correlation coefficient 'r' is independent of the change of origin and scale.

Also we note that correlation coefficient is a dimensionless number independent of the units in which variables X and Y are measured.

Example 21.68: In an experiment to determine the relationship between force on a wire and the resulting extension, the following data is obtained:

Force (N) :	10	20	30	40	50	60	70
Extension (mm.) :	0.22	0.40	0.61	0.85	1.20	1.45	1.70

Also, $\sigma_y^2 = E[Y - E(Y)]^2 = \frac{a^2}{b^2} E[X - E(X)]^2 = \frac{a^2}{b^2}\sigma_x^2$, which gives, $\sigma_y = \left|\frac{a}{b}\right|\sigma_x$.

Hence, $$r_{xy} = \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y} = -\frac{a}{b}\bigg/\left|\frac{a}{b}\right|$$

$$= \begin{cases} +1, & \text{if } a \text{ and } b \text{ are of the opposite signs.} \\ -1, & \text{if } a \text{ and } b \text{ are of the same signs.} \end{cases}$$

Example 21.72: The joint probability distribution of (X, Y) is given below. Find the correlation coefficient between X and Y.

$Y \backslash X$	-1	$+1$
0	1/8	3/8
1	1/4	1/4

Solution: Let $p(x)$ and $q(y)$ be the marginal probabilities for X and Y. We form the following table:

$Y \backslash X$	-1	$+1$	$q(y)$
0	1/8	3/8	1/2
1	1/4	1/4	1/2
$p(x)$	3/8	5/8	1

Thus
$$E(X) = \sum xp(x) = -3/8 + 5/8 = 1/4,$$
$$E(X^2) = \sum x^2 p(x) = 3/8 + 5/8 = 1,$$
$$E(Y) = \sum yq(y) = 0 + 1/2 = 1/2,$$
$$E(Y^2) = \sum y^2 q(y) = 0 + 1/2 = 1/2,$$

and,
$$E(XY) = \sum xyp(x, y) = 0 + 0 + \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) = 0.$$

Hence,
$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = -1/8,$$
$$\sigma_x^2 = E(X^2) - [E(X)]^2 = 1 - \frac{1}{16} = 15/16,$$

and,
$$\sigma_y^2 = E(Y^2) - [E(Y)]^2 = 1/2 - 1/4 = 1/4.$$

Therefore,
$$r_{xy} = \frac{-1/8}{\sqrt{15/16}\,.\sqrt{1/4}} = -0.2582.$$

21.13.2 Spearman's Rank Correlation Coefficient

Let us suppose that a group of n individuals is arranged in order of merit (rank) regarding two characteristics A and B. These ranks in respect of the two characteristics, in general, will be different.

Let x_i, y_i; $i = 1, 2, ..., n$ be the ranks of the ith individual in respect to characteristics A and B respectively. The Spearman's rank correlation coefficient, denoted by ρ, is simply Pearson's coefficient of correlation calculated for the ranks x_i's and y_i's.

Assuming that no two individuals have the same rank, each of the two variables x and y takes the values $1, 2, ..., n$.

Hence, $$\bar{x} = \bar{y} = \frac{n+1}{2},$$

and, $$\Sigma (x_i - \bar{x})^2 = \Sigma (y_i - \bar{y})^2 = \Sigma y_i^2 - n\bar{y}^2$$

$$= \frac{n(n+1)(2n+1)}{6} - n\left(\frac{n+1}{2}\right)^2 = \frac{1}{12}(n^3 - n).$$

Let $$d_i = x_i - y_i = (x_i - \bar{x}) - (y_i - \bar{y}), \text{ therefore}$$

$$\frac{1}{n}\Sigma d_i^2 = \frac{1}{n}\Sigma (x_i - \bar{x})^2 + \frac{1}{n}\Sigma (y_i - \bar{y})^2 - \frac{2}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y}).$$

It gives $$2\,\mathrm{Cov}(X, Y) = \frac{2}{n}\Sigma (x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{n}\Sigma (x_i - x)^2 + \frac{1}{n}\Sigma (y_i - \bar{y})^2 - \frac{1}{n}\Sigma d_i^2$$

$$= \frac{2}{n}\cdot\frac{1}{12}(n^3 - n) - \frac{1}{n}\Sigma d_i^2$$

or, $$\mathrm{Cov}(X, Y) = \frac{1}{12}(n^2 - 1) - \frac{1}{2n}\Sigma d_i^2.$$

Also, $$\sigma_x\sigma_y = \sqrt{\frac{1}{n}\Sigma (x_i - \bar{x})^2}\cdot\sqrt{\frac{1}{n}\Sigma (y_i - \bar{y})^2}$$

$$= \frac{1}{n}\cdot\frac{1}{12}(n^3 - n) = \frac{1}{12}(n^2 - 1).$$

Hence the correlation coefficient between the two characteristics A and B is

$$\rho = \frac{\mathrm{Cov}(x, y)}{\sigma_x\,\sigma_y} = 1 - \frac{6\Sigma d_i^2}{n(n^2 - 1)}, \qquad ...(21.98)$$

called the *Spearman's rank correlation coefficient.*

Remarks

1. If any two or more individuals are bracketed, then these individuals are given the common rank equal to the average of the ranks they would have assumed in case of marginal differences. The formula (21.98) is corrected by adding the factor $\frac{m(m^2-1)}{12}$ to $\Sigma\, d^2$, where m is the number of items tied up. This factor is to be compensated for each tied group in both the x's and y's. The procedure is illustrated in Example (21.74).

2. Spearman's rank correlation coefficient ρ also lies between -1 to 1. ρ is maximum when $\Sigma\, d_i^2$ is minimum, that is, zero. In this case, $x_i = y_i$; $i = 1, 2, ..., n$. Hence, *maximum value of* ρ *is* 1. Further ρ is minimum when x_i and y_i have opposite ranks, that is, $x_i + y_i = n + 1$; $i = 1, 2, ..., n$. In this case,

$$\begin{aligned}\Sigma\, d_i^2 &= \Sigma\,(x_i - y_i)^2 \\ &= \Sigma\,[2x_i - (n+1)]^2 \\ &= 4\Sigma\, x_i^2 + n(n+1)^2 - 4(n+1)\,\Sigma\, x_i \\ &= 4\,.\frac{n(n+1)\,(2n+1)}{6} + n(n+1)^2 - 2n(n+1)^2 \\ &= n(n+1)\left[\frac{2(2n+1)}{3} - (n+1)\right] = \frac{n(n^2-1)}{3}\end{aligned}$$

Hence,
$$\text{min. } \rho = 1 - \frac{6\,\frac{n(n^2-1)}{3}}{n(n^2-1)} = 1 - 2 = -1.$$

Thus ρ lies between -1 and 1.

Example 21.73: Ten students got the following marks in mathematics and economics in end-sem examination. Calculate the rank correlation coefficient.

Students	:	1	2	3	4	5	6	7	8	9	10
Marks in Economics	:	78	36	98	25	75	82	90	62	65	39
Marks in Mathematics	:	81	51	91	60	68	62	86	58	53	47

Solution: Alloting the ranks to the students, we have

Students	:	1	2	3	4	5	6	7	8	9	10
Rank in Eco.	:	4	9	1	10	5	3	2	7	6	8
Rank in Maths.	:	3	9	1	6	4	5	2	7	8	10
Difference (d_i)	:	1	0	0	4	1	−2	0	0	−2	−2
d_i^2	:	1	0	0	16	1	4	0	0	4	4

We have, $\Sigma\, d_i^2 = 30$

Therefore,
$$\rho = 1 - \frac{6\,\Sigma\, d_i^2}{n(n^2-1)} = 1 - \frac{180}{10(100-1)} = 0.8182,$$

which shows a good positive correlation between the two characteristics.

Example 21.74: Obtain the rank correlation coefficient for the following data:

x	:	68	64	75	50	64	80	75	40	55	64
y	:	62	58	68	45	81	60	68	48	50	70

Solution: We form the following table:

x	*y*	*Rank x*	*Rank y*	*d = x – y*	d^2
68	62	4	5	–1	1
64	58	6	7	–1	1
75	68	2.5	3.5	–1	1
50	45	9	10	–1	1
64	81	6	1	5	25
80	60	1	6	–5	25
75	68	2.5	3.5	–1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	70	6	2	4	16
				Total: 0	72

Due to common ranks in the x-series, the average rank 2.5 has been assigned twice and the average rank 6 has been assigned thrice. Similarly in y-series, the average rank 3.5 has been assigned twice. Hence, the correction in $\Sigma\, d^2$ due to common ranks corresponds to m = 2, 3 and 2 and the correction factors is $\dfrac{m(m^2-1)}{12}$.

$$\text{Thus the total correction} = \frac{2(4-1)}{12} + \frac{3(9-1)}{12} + \frac{2(4-1)}{12} = \frac{1}{2} + 2 + \frac{1}{2} = 3.$$

$$\text{Hence,} \qquad \rho = 1 - \frac{6(\Sigma\, d^2 + 3)}{n(n^2-1)} = 1 - \frac{6(72+3)}{10 \times 99} = 0.545.$$

21.13.3 Regression

In regression analysis one of the two variables, say x is regarded as an independent variable, the other variable y as the dependent variable which may be random in nature. We are interested in the dependence of y on x. For example, dependence of blood pressure (y) on the age (x) of a person, or the dependence of height of son (y) on the height of father (x), etc. In general, we specify $x_1, x_2, ..., x_n$, and then observe the corresponding values $y_1, y_2, ..., y_n$ of the r.v. Y,. so that we get a bivariate sample (x_i, y_i); $i = 1, 2, ..., n$. We are interested in finding a mathematical measure of the average relationship between the two (or, more in case of multivariate data) variables in terms of the original units of the data.

Linear Regression: If the variables in the bivariate distribution (x_i, y_i); $i = 1, 2, ..., n$, are related, then the points in the scatter diagram will converge around some curve, called the *curve of*

regression. In case the curve happens to be a straight line, the regression is said to be *linear* and the line is called the *line of regression*. *"It is the line which gives the best estimate of the dependent variable for any specific value of the independent variable"*. The best-fit line is obtained by the principle of least squares, which consists in minimising the sum of the squares of the deviations of the actual values of y from their estimated values as given by the line of best-fit.

Let the line of regression of Y on x be

$$Y = a + bx.$$

Then the error of estimate for $Y = y_i$, refer Fig. 21.13, is

$$P_iM_i = y_i - (a + bx_i).$$

According to the principle of least squares, we need to find a and b so that

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

is minimum.

Thus the normal equations for estimating a and b are

$$\sum y_i = na + b\sum x_i \qquad ...(21.99)$$

and,

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \qquad ...(21.100)$$

Dividing (21.99) by n, we obtain

$$\bar{y} = a + b\bar{x}. \qquad ...(21.101)$$

Thus the line of regression passes through $(\bar{x}, \bar{y})$.

Next, dividing (21.100) by n, we obtain

$$\frac{1}{n}\sum x_i y_i = \frac{a}{n}\sum x_i + \frac{b}{n}\sum x_i^2$$

or,

$$[\text{Cov}(x, y) + \bar{x}\,\bar{y}] = a\bar{x} + b[\sigma_x^2 + (\bar{x})^2], \qquad ...(21.102)$$

since $\text{Cov}(x, y) = \frac{1}{n}\sum x_i y_i - \bar{x}\,\bar{y}$, and $\sigma_x^2 = \frac{1}{n}\sum x_i^2 - (\bar{x})^2$.

To obtain b, multiply (21.101) by $\bar{x}$ and subtract from (21.102), we obtain b, *the slope of the line of regression of y on x*, as

$$b = \frac{\text{Cov}(x, y)}{\sigma_x^2}. \qquad ...(21.103)$$

Hence the *equation of the line of regression of y on x* is

$$y - \bar{y} = b(x - \bar{x}) = \frac{\text{Cov}(x, y)}{\sigma_x^2}(x - \bar{x})$$

or,

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}), \qquad ...(21.104)$$

where r is the coefficient of correlation between x and y.

On interchanging the variables, we arrive at the *equation of the line of regression of x on y* given by

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}). \qquad ...(21.105)$$

The two coefficients b_{yx} and b_{xy}, given respectively by

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} \quad \text{and} \quad b_{xy} = r\frac{\sigma_x}{\sigma_y}, \qquad ...(21.106)$$

are called the *regression coefficients* of y on x and of x on y respectively.

We observe that $\qquad r = \pm\sqrt{b_{yx}b_{xy}}$.

Thus the coefficient of correlation r between the two variables X and Y is the geometric mean of the two coefficients of regression and the sign of r is taken as the common sign of b_{yx} and b_{xy}.

Also we should note that the regression coefficients are independent of the change of origin but depends upon the change of scale.

Angle between two lines of regression

If θ is the acute angle between the two regression lines (21.104) and (21.105), then

$$\tan\theta = \left|\frac{r\dfrac{\sigma_y}{\sigma_x} - \dfrac{\sigma_y}{r\sigma_x}}{1 + r\dfrac{\sigma_y}{\sigma_x}.\dfrac{\sigma_y}{r\sigma_x}}\right| = \frac{1-r^2}{|r|}\,\frac{\sigma_x\,\sigma_y}{\sigma_x^2 + \sigma_y^2} \qquad ...(21.107)$$

If $r = 0$, then $\tan\theta = \infty$, that is, $\theta = \pi/2$. *Thus, if two variables are uncorrelated, then the regression lines are perpendicular to each other.*

If $r = \pm 1$, then $\tan\theta = 0$, that is, $\theta = 0$. *Thus, in case of perfect correlation, positive or negative, the two regression lines coincide.*

Remark: There are always two regression lines, one of y on x and other of x on y. The line of regression of y on x gives the best estimate of y for given specific value of x and is used when x is an independent variable and y is a dependent variable. If the situation is otherwise, y being independent and x dependent, then the line of regression of x on y is used.

Example 21.75: The following are measurements of the air velocity(x) and evaporation coefficient (y) of burning fuel droplets in an impulse engine. Find the coefficient of correlation r and the line of regression of y on x. Also estimate the evaporation coefficient when the air velocity is 210 cm/sec.

Air velocity (cm/sec) (x)	:	20	60	100	140	180	220	260	300	340	380
Evaporation coefficient (mm^2/sec) (y)	:	0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.36	1.17	1.65

On the basis of the data available what should be the expected air velocity in case evaporation of 1.50 mm^2/sec. is needed?

Solution: For this set of observations

$$n = 10, \qquad \Sigma x_i = 2000, \qquad \Sigma x_i^2 = 532{,}000$$

$$\Sigma y_i = 8.35, \qquad \Sigma x_i y_i = 2175.40, \qquad \Sigma y_i^2 = 9.11.$$

Thus, $\bar{x} = 200, \qquad \bar{y} = 0.835,$

and, $$\sigma_x^2 = \frac{1}{n}\Sigma x_i^2 - (\bar{x})^2 = 53200 - 40000 = 13200$$

Therefore, $\sigma_x = 114.89.$

Also $$\sigma_y^2 = \frac{1}{n}\Sigma y_i^2 - (\bar{y})^2 = 0.911 - 0.6997 = 0.214,$$

Therefore, $\sigma_y = 0.463.$

$$\text{Cov}(x, y) = \frac{1}{n}\Sigma x_i y_i - (\bar{x})(\bar{y}) = 217.54 - (200)(0.835) = 50.54.$$

Thus, $$r = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = \frac{50.54}{(114.89)(0.463)} = 0.95.$$

Also, $$b_{yx} = r\frac{\sigma_y}{\sigma_x} = (0.95)\frac{0.463}{114.89} = 0.00383.$$

Thus the equation of the line of regression of y on x is

$$y - 0.835 = 0.00383\,(x - 200)$$

or, $$y = 0.00383\,x + 0.069.$$

When $x = 210$ cm/sec, then estimated y is, $y = 0.00383\,(210) + 0.0069 \simeq 0.8733$ mm/sec^2

To find the expected air-velocity (x) corresponding to the evaporation coefficient (y) of 1.50 mm^2/sec, we need to find the regression equation of x on y.

We have, $$b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.95.\frac{114.89}{0.463} = 235.735$$

Hence, the regression line of x on y is

$$x - 200 = (235.735)(y - 0.835)$$

or, $$x = 235.735\,y + 3.161.$$

Thus, the expected air-velocity corresponding to evaporation coefficient of 1.50 mm^2/sec is $x = (235.735)(1.50) + 3.61 \approx 357.21$ cm/sec.

Example 21.76: Obtain the equation of the line of regression of y on x for the following data. Also estimate y when $x = 32$.

x	:	23	27	28	28	29	30	31	33	35	36
y	:	18	20	22	27	21	29	27	29	28	29

Solution: Let $u = x - 29$, $v = y - 25$. Refer Example (21.69), we have

$$n = 10, \quad \sum u = 10, \quad \sum u^2 = 148, \quad \sum v = 0$$

$$\sum v^2 = 164, \quad \sum uv = 123, \text{ and } \quad r_{xy} = r_{uv} = 0.86$$

Also,

$$\bar{x} = a + \bar{u} = 29 + \frac{10}{10} = 30.$$

$$\bar{y} = b + \bar{v} = 25 + 0 = 25.$$

$$\sigma_x = \sigma_u = \sqrt{\frac{148}{10} - (1)^2} = 3.71.$$

$$\sigma_y = \sigma_v = \sqrt{\frac{164}{10} - 0} = 4.05.$$

Hence, the line of regression of y on x is

$$y - 25 = (0.86)\frac{4.05}{3.71}(x - 30)$$

or,

$$y = 0.94x - 3.2.$$

Thus, the estimated value of y for $x = 32$ is

$$y = (0.94)(32) - 3.2 = 26.88.$$

Example 21.77: Can $y = 5 + 2.8\,x$ and $x = 3 - 0.5y$ be the estimated regression equations of y on x and x on y respectively? Explain.

Solution: In case the given equations happen to be the equations of the regression lines, then the coefficient of regression of y on x is $b_{yx} = 2.8$, and the coefficient of regression of x on is $b_{xy} = -0.5$, which is not possible since the two coefficients of regression are always of the same sign that of the sign of Cov (x, y). Hence, the given equations cannot be the regression equations.

EXERCISE 21.8

1. In an experiment to determine the relationship between the current flowing in an electrical circuit and the voltage applied, the results obtained are

Current (mA) :	5	11	15	19	24	28	33
Applied voltage (V) :	2	4	6	8	10	12	14

Find the Karl-Pearson's coefficient of correlation.

2. A gas is being compressed in a closed cylinder and the values of pressures and corresponding volumes at constant temperature are as shown:

Pressure (kP_a) :	160	180	200	220	240	260	280	300
Volume (m^3) :	0.034	0.036	0.030	0.027	0.024	0.025	0.020	0.019

Find the coefficient of correlation for these values.

3. The following marks have been obtained by a class of students in statistics

Paper I :	45	55	56	58	60	65	68	70	75	80	85
Paper II :	56	50	48	60	62	64	65	70	74	82	90

Compute the coefficient of correlation for the above data.
Find also the equations of the lines of regression.

4. For the following data, compute the coefficient of correlation between X and Y.

		X-series	Y-series
No. of items	:	15	15
Arithmetic mean	:	25	18
Sum of square of deviations from mean	:	136	138

Also the sum of products of deviations of X and Y from their respective arithmetic means is 122.

5. For the bivariate probability distribution given below, find the coefficient of correlation between X and Y.

Y \ X	–1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

6. Calculate rank correlation coefficient for the following measurements of the air velocity (x) and evaporation coefficient (y) of burning fuel droplets in an impulse engine. Compare it with Karl-Pearson's coefficient of correlation.

x(cm/sec)	:	20	60	100	140	180	220	260	300	340	280
y(mm^2/sec)	:	0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.36	1.17	1.65

7. Ten competitors in a musical contest were ranked by the three judges A, B and C in the following order:

Ranks by A:	1	6	5	10	3	2	4	9	7	8
Ranks by B:	3	5	8	4	7	10	2	1	6	9
Ranks by C:	6	4	9	8	1	2	3	10	5	7

Discuss which pair of judges has the nearest approach to common likings in music.

8. Calculate the rank correlation coefficient for the following data:

x	:	65	63	67	64	68	62	70	66	68	67	69	71
y	:	68	66	68	65	69	66	68	65	71	67	68	70

9. If x_1, x_2, x_3 are uncorrelated variables each having the same S.D.'s, obtain the coefficient of correlation between $x_1 + x_2$ and $x_2 + x_3$.

10. Find the coefficient of correlation between the number of heads and number of tails obtained in n throws of a coin.

12. Find the coefficient of correlation from the following table giving the ages of husbands(x) and wives(y) in case of 100 couples surveyed,

22 CHAPTER
Sampling Distributions and Hypothesis Testing

"The sample statistics describe the sample and are used to make inference about the sampled population parameters. In hypothesis testing our primary concern is to develop a procedure for determining whether or not the values of the random sample from the population are consistent with the hypothesis. Distribution of the sample statistic facilitates to formulate that procedure."

22.1 BASIC CONCEPTS

In any particular study the number of observations recorded may be finite or infinite. For example, number of defective screws in a box of 1000 results in a finite number of observations, while if we could toss a pair of dice indefinitely and record the total obtained, then we obtain an infinite set of observations. A *population* consists of the totality of the observations under study. In case the number of observations are finite, population is called *finite population* otherwise *infinite population.* When it is not desirable to take into account all the observations, then we take a finite subset of the population called *sample.* In this chapter, we focus on sampling from population and study constants of the sample drawn called *statistics* like, sample mean $(\overline{x})$, sample variance (s^2), etc. and see how the information drawn from the sample is utilized to draw some conclusion about the population constants like, mean (μ), variance (σ^2), etc. called *parameters.*

The main *advantages of sampling* over complete enumeration consist of reduced cost, greater speed and scope and sometimes even better precision. When testing is destructive in nature, then sampling becomes necessary.

Methods of sampling: Some of the commonly employed methods of sampling are:

Random sampling: If each unit of the population has the same chance of being selected in the sample then sampling is said to be random sampling. Suppose we take a sample of size n from a population of finite size N, then in case of random sampling each of the $^{N}C_n$ samples has the same probability, that is, $1/^{N}C_n$ of being selected.

The simplest method of drawing a random sample is the lottery method, assigning numbers 1 to N to each unit of the population; writing these numbers on N identical slips; putting these slips in a box and then drawing n slips one by one from this well-shuffled lot. Then n units, corresponding to the numbers on the slips drawn constitute the random sample. Since this method is always not very practical, a simpler and more reliable method is the use of random numbers. *Random numbers* are the digits generated so that values 0 to 9 occur randomly with equal frequency. These numbers can be generated by computer, or alternatively, available from *random numbers tables* due to L.H.C. Tippett. These tables consist of 10400 four-digit numbers giving in all 41600 digits taken from British Census Reports. Random numbers help in obtaining samples that are in fact random samples.

Simple random sampling: Simple random sampling is random sampling in which each unit of the population has an equal probability 'p' of being included in the sample and this probability is independent of the previous drawings. Thus random sampling becomes simple, if either the units are drawn with replacement, or when the population is infinite. A simple sample of size n from a population may be identified with a series of n Bernoulli trials with constant probability 'p' of success for each trial.

Stratified sampling: In case the population is heterogeneous, then it is divided into homogeneous stratas (groups) of various sizes. Then units are sampled at random from each of these strata according to the relative importance of the stratas in the population. The sample drawn thus will be more representative than the simple random sample, since each strata will be represented in the sample drawn.

Systematic sampling: Let there be population with $N = nk$ ordered units from 1 to N. In systematic sampling if we are to draw a sample of size n, then we draw a unit at random from the first k ordered units and, then every kth unit is drawn to form the sample.

Purposive sampling: Here the units are selected with definite purpose in view. Usually, the selected units do not form a representative sample of the population and yield results which are generally biased.

In general, not all sampling plans involve random selection but any sampling plan used for drawing inferences must involve randomization.

22.2 STATISTICS AND SAMPLING DISTRIBUTIONS

The numerical descriptive measures calculated from the sample are called *statistics* and the numerical descriptive measures of the population, (generally unknown), are called *parameters*. These statistics vary for each different random sample selected and hence they are random variables. The probability distributions for statistics e.g., for sample mean, sample variance, etc. are called *sampling distributions.*

For example, consider a population consisting of 5 numbers 3, 5, 7, 9, 11. If a random sample of size $n = 2$ is selected without replacement, then we can find the sampling distribution of the sample mean $\bar{x}$ as follow.

There are 10 possible equally likely random samples of size $n = 2$. The values of $\bar{x}$ for random sampling when $n = 2$ and $N = 5$ are tabulated below.

Sample	*Sample units*	*Sample mean,* $\bar{x}$
1	3, 5	4
2	3, 7	5
3	3, 9	6
4	3, 11	7
5	5, 7	6
6	5, 9	7
7	5, 11	8
8	7, 9	8
9	7, 11	9
10	9, 11	10

Hence, sampling distribution of the sample mean $\bar{x}$ is

$$\begin{array}{rccccccc} \bar{x} = & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ f(\bar{x}) = & 1 & 1 & 2 & 2 & 2 & 1 & 1 \\ p(\bar{x}) = & 0.1 & 0.1 & 0.2 & 0.2 & 0.2 & 0.1 & 0.1 \end{array}$$

$$\text{Population mean, } \mu = \frac{3+5+7+9+11}{5} = 7$$

$$\text{Population variance, } \sigma^2 = \frac{(-4)^2+(-2)^2+0+(2)^2+(4)^2}{5} = 8$$

$$\text{Mean of 'sample means'} = 0.4 + 0.5 + 1.2 + 1.4 + 1.6 + 0.9 + 1.0 = 7.0$$

$$\begin{aligned} \text{Variance of 'sample means'} &= (-3)^2(0.1) + (-2)^2(0.1) + (-1)^2(0.2) + 0 + (1)^2(0.2) + (2)^2(0.1) + (3)^2(0.1) \\ &= 0.9 + 0.4 + 0.2 + 0.2 + 0.4 + 0.9 = 3.0 \end{aligned}$$

We observe that mean of the sample means is the same as the population mean but variance of the sample means is not the same as the population variance.

The standard deviation of sampling distribution of a statistic is called its *standard error* (S.E.). In this case S.E. is $\sqrt{3}$.

Normally, we use statistical theory to derive sampling distribution of a statistic or use simulation to derive the sampling distribution empirically.

An important result which describes the sampling distribution of a statistic which are sum or averages of the sample observations is the *Central limit theorem* stated as below:

Theorem 22.1 (Central limit theorem): *If random samples of n observations are drawn from a non-normal population with finite mean μ and standard deviation σ, then when n is large, the sampling distribution of the sample mean $\bar{x}$ is approximately normally distributed with mean μ and S.E. $\sigma/\sqrt{n}$.*

This approximation becomes more accurate as *n* becomes large.

The central limit theorem has important contribution in statistical inference since many estimators that are used to make inference about population parameters are sum or averages of the

sample observations, and when the sample size n is large then these estimators can be approximated as normal variates.

In case the population itself is normal, then sampling distribution of $\bar{x}$ is always normal, irrelevant of the size n of the sample selected. But when the population is skewed then the sample size n must be large, say $n > 30$ to approximate the distribution of $\bar{x}$ as normal.

22.2.1 The Sampling Distribution of the Sample Mean

If the population mean μ is unknown, then the statistic sample mean $\bar{x}$, in general, is chosen as the natural estimate of the population mean. The following theorem gives sampling distribution of the sample mean $\bar{x}$.

Theorem 22.2 (Sampling distribution of the sample mean): *If a random sample of size n is selected from a population with mean μ and S.D. σ, then the sampling distribution of the sample mean $\bar{x}$ will have mean μ and standard error (S.E.)* $\sigma/\sqrt{n}$.

Proof: Let $x_1, x_2, ..., x_n$ be a random sample of size n drawn from a population of size N with mean μ and variance σ^2. Then the sample mean, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and the sample variance, $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Consider
$$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu \qquad ...(22.1)$$

$$\text{var}(\bar{x}) = \text{var}\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) = \frac{1}{n^2}[\text{var}(x_1) + \text{var}(x_2) + \ldots + \text{var}(x_n)],$$

since x_i's are independent, thus the co-variances terms are absent. Hence

$$\text{var}(\bar{x}) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \qquad ...(22.2)$$

and,
$$\text{S.E.}(\bar{x}) = \sigma/\sqrt{n}. \qquad ...(22.3)$$

Hence $\bar{x}$ is distributed with mean μ and S.E. $\sigma/\sqrt{n}$.

In case the sampled population is normal, the distribution of $\bar{x}$ will be exactly normal irrespective of the size n, however, if the population is non-normal, then the distribution of $\bar{x}$ will be approximately normal for large n by central limit theorem. Thus the statistic z is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1) \qquad ...(22.4)$$

Remark: A statistic θ is called an *unbiased estimator* of a polulation parameter γ, if $E(\theta) = \gamma$. Since, $E(\bar{x}) = \mu$, thus *sample mean $\bar{x}$ is an unbiased estimate of the population mean* μ. But sample variance, $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is not an unbiased estimate of the population variance as shown next.

We have, $E(s^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right) = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\bar{x})^2\right]$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i^2) - E(\bar{x}^2) \qquad ...(22.5)$$

Now $E(x_i^2) = \text{var}(x_i) + [E(x_i)]^2 = \sigma^2 + \mu^2,$

and, $E(\bar{x}^2) = \text{var}(\bar{x}) + [E(\bar{x})]^2 = \frac{\sigma^2}{n} + \mu^2,$ using (22.1) and (22.2).

Using these in (22.5), we obtain

$$E(s^2) = \frac{1}{n}\sum_{i=1}^{n}(\sigma^2 + \mu^2) - \left(\frac{\sigma^2}{n} + \mu^2\right) = \left(1 - \frac{1}{n}\right)\sigma^2 \neq \sigma^2.$$

However, if we define $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, then

$$E(S^2) = E\left(\frac{n}{n-1}s^2\right) = \frac{n}{n-1}E(s^2) = \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2. \qquad ...(22.6)$$

Hence, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ *is an unbiased estimate of the population variance* σ^2.

22.2.2 The Sampling Distribution of the Sample Proportion

In some practical situations we need to estimate the proportion 'p' of people in the population who have a specified characteristic say smoking, computer literacy, etc. If x out of the n sampled people have this characteristic, then the sample proportion $\hat{p} = x/n$ can be taken as an estimate of the population proportion p. We observe that the distribution of the random variable x is binomial with mean np and S.D. $\sqrt{npq}$, and thus $\hat{p} = \frac{x}{n}$ will also be distributed like a binomial variate with mean as

$$E(\hat{p}) = E\left(\frac{x}{n}\right) = \frac{1}{n}E(x) = p, \qquad ...(22.7)$$

variance as

$$\text{var}(\hat{p}) = \text{var}\left(\frac{x}{n}\right) = \frac{1}{n^2}\text{var}(x) = \frac{pq}{n} \qquad ...(22.8)$$

and, standard error as

$$\text{S.E. }(\hat{p}) = \sqrt{\frac{pq}{n}}, \qquad \text{...(22.9)}$$

where $q = 1 - p$.

Further, since the binomial distribution can be approximated to normal for large n, thus the statistic z given by

$$z = \frac{\hat{p} - p}{\sqrt{pq/n}} \qquad \text{...(22.10)}$$

will be a standard normal variate for large n, that is, $z \sim N(0, 1)$.

Example 22.1: An electrical firm manufactures light bulbs that have burning life normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an average burning life of less than 775 hours.

Solution: Let $\bar{x}$ denote the average burning life of 16 bulbs, then $\bar{x}$ is a normal variate with mean 800 hrs and S.E. = $40/\sqrt{16}$ = 10. Thus

$$z = \frac{\bar{x} - 800}{10} \sim N(0, 1).$$

Hence, $P(\bar{x} < 775) = P(z < -2.5) = P(z > 2.5) = 0.5 - P(0 < z < 2.5) = 0.5 - 0.4938 = 0.0062$.

Example 22.2: The duration of Alzheimer's disease from the appearance of symptoms until death, is distributed with an average of 9 years and a S.D. of 4 years. The medical records of 36 randomly selected deceased patients from a large medical database has been taken. Find the approximate probability that average duration lies within 7 and 11 years.

Solution: Let $\bar{x}$ be the average duration of survival after the appearance of symptoms. Since the sample size n is 36, $\bar{x}$ can be approximated as a normal variate with mean $\mu = 9$ and standard error $= \sigma/\sqrt{n} = 4/\sqrt{36} = 2/3$. Thus

$$z = \frac{\bar{x} - 9}{2/3} \sim N(0, 1).$$

Hence, $P(7 < \bar{x} < 11) = P(-3 < z < 3) = 2P(0 < z < 3) = 2(0.4987) = 0.9974$

Example 22.3: A random sample of 100 students was taken from a campus and 12 were found to be smokers. Estimate the proportion of smokers in the campus as well as the S.E. of the estimate. Find the almost certain limits to the percentage of smokers in the campus.

Solution: The proportion of smokers in the sample is

$$\hat{p} = \frac{12}{100} = 0.12, \quad \hat{q} = 0.88$$

Hence the S.E. $(\hat{p}) = \sqrt{\dfrac{(0.12)\,(0.88)}{100}} = 0.0325$, using (22.9).

Hence, the proportion of smokers lies certainly between

$$\hat{p} \pm 3(\text{S. } E.) = 0.12 \pm 3(0.0325) = 0.12 \pm 0.0975,$$

that is between 0.0225 and 0.2175. Therefore the percentage of smokers almost certainly lies between 2.25 and 21.75.

EXERCISE 22.1

1. A soft drink machine is being regulated so that the amount of drink dispensed averages 240 ml. with a S.D. 15 ml. The machine is checked periodically by taking a sample of 40 drinks and if the mean amount $\bar{x}$ for the sample taken lies within $E(\bar{x}) \pm 2$ S.E. $(\bar{x})$, the machine is certified O.K., otherwise is rectified. An apprentice from the company found the mean of 40 drinks to be 236 ml and certified O.K. Was that a reasonable decision? Justify your answer.
2. A random sample of 500 fuses was taken from a large consignment and 65 were found to be defective. Show that the percentage of defectives in the consignment almost certainly lies between 8.5 and 17.5.
3. A coin is tossed 1000 times and the head comes out 550 times. Can the deviation from expected value be due to fluctuations of sampling?
4. A large batch of electric bulbs have a mean time to failure of 800 hours and the S.D. of 60 hours. For a random sample of 64 electric bulbs determine the probability that mean time to failure will be

   (a) less than 785 hours, (b) more than 820 hours.
5. The contents of a consignment of 1200 tins of a product have a mean mass of 5040 gm with a S.D. of 2.3 gm. Find the probability that a random sample of 40 tins drawn from the consignment will have a combined mass of

   (a) less than 20.13 kg, (b) between 20.13 kg and 20.17 kg, and(c) more than 20.17 kg.
6. It has been observed that almost 75% of customers visiting a fabric mall prefer natural fabrics in comparison to man-made fabrics. A random sample of 200 customers has been selected and the number who like natural fabrics is recorded.

   (a) What is the approximate sampling distribution for the sample proportion $\hat{p}$?

   (b) What is the probability that the sample proportion is greater than 80%?

   (c) Within what limits the sample proportion are expected to lie about 95% of the time?

22.3 NULL AND ALTERNATIVE HYPOTHESIS. TYPES OF ERRORS AND LEVEL OF SIGNIFICANCE

An important aspect of sampling theory is to make decision about the parameter value. The tests of hypothesis enable us to decide on the basis of the statistic obtained, that whether the deviation between the observed and the theoretical value is significant or might be attributed to fluctuations

of sampling. Since for large n the sampling distribution of the statistic under study can be approximated to normal, so for large sample testing normal distribution is applied. However, in case of small sample testing we employ specific variates like t, χ^2, F, etc.

22.3.1 Null and Alternative Hypotheses

A 'statistical hypothesis' is an assertion concerning one or more populations. A hypothesis we wish to test, is called the 'null hypothesis' and is denoted by H_0; and any hypothesis, complementary to the null hypothesis, is called an 'alternative hypothesis' and is usually denoted by H_1. The null hypothesis H_0 is usually a hypothesis of 'no-difference' and is tested for possible rejection under the assumption that it is true.

For example, if we want to test that average daily wages of workers in a construction company is different from Rs. 170, the national average, then we can set up the null hypothesis as

$$H_0 : \mu = 170$$

and, the alternative hypothesis could be any of

(a) $H_1 : \mu \neq 170$ (b) $H_1 : \mu < 170$ (c) $H_1 : \mu > 170$.

The alternative hypothesis, (a) is known as *two-tailed alternative*; (b) is known as *left-tailed alternative*; and (c) as *right-tailed alternative*.

The hypothesis (a) is *composite alternative*, while hypotheses (b) and (c) are *simple alternatives*.

22.3.2 Acceptance and Rejection Regions

The decision to reject or accept the null hypothesis is based on the information contained in a sample drawn from the population under study. On the basis of the data in the sample a *test statistic* is formulated and using this test statistic a probability value is calculated (generally, obtained from the tables available). On the basis of these measures obtained the hypothesis H_0 is rejected or accepted. Now the important question arises: *How to decide whether to reject or accept H_0?* This is answered as follows.

Since, our decision is based on the value of the test statistic obtained, thus entire set of values that the test statistic may attain is divided into two regions, the acceptance region and the rejection region.

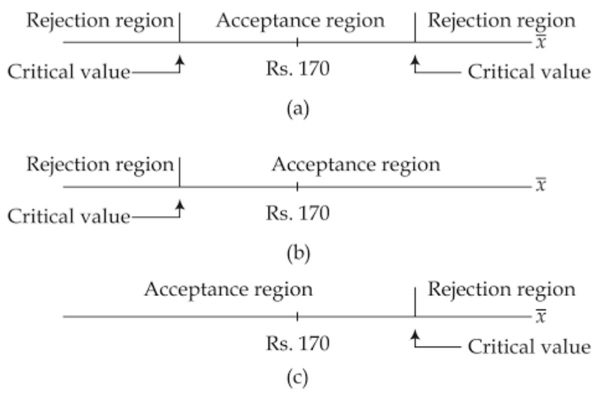


Fig. 22.1

The region consisting of the values which support the null hypothesis, leading to acceptance of H_0, is called the *acceptance region*, and the region consisting of values which support the alternative hypothesis, leading to rejection of H_0, is called the *rejection region* or the *critical region*. The value (s) that separate the acceptance and rejection region is (are) called the *critical value* (s).

For example, in case we wish to test the null hypothesis $H_0 : \mu = 170$ against the alternative $H_1 : \mu \neq 170$, then the acceptance and rejection regions are as shown in Fig. 22.1a.

In case the alternative is $H_1 : \mu < 170$, or $H_1 : \mu > 170$, then acceptance and rejection regions are as shown in Figs. 22.1b and 22.1c, respectively.

The type of test in case of (a) is called *two-tailed test* and, in case of (b) and (c) is called *left-tailed test* and *right-tailed test,* (jointly, as *single-tailed tests*), respectively.

22.3.3 Types of Errors and Level of Significance

The decision procedure described above can lead to either of the following two types of errors:

Type I error : Rejection of the null hypothesis H_0 when it is true (*Rejection error*).

Type II error : Acceptance of the null hypothesis H_0 when it is false (*Acceptance error*).

The probability of Type I error, that is,

P\{Rejection of H_0 when it is true\} = P\{Reject $H_0 \mid H_0$\},

is called the *level of significance,* or *size of the test* and is denoted by α.

The probability of Type II error, that is,

P\{Acceptance of H_0 when it is false\} = P\{Accept $H_0 \mid H_1$\}

is denoted by β.

Remarks:

1. In statistical quality control Type I error amounts to rejecting a lot when it is good and Type II error may be regarded as accepting the lot when it is bad, thus α and β are often referred to as *producer's risk* and *consumer's risk* respectively.

2. For a fixed sample size, a decrease in the probability of one type error will usually result in an increase in the probability of the other type of error. Both types of errors can be reduced only by increasing the sample size n.

3. For applying the test of significance, the level of significance, that is, the size of Type I error is kept fixed normally at 5% or 1% and the sampling is so designed that for given α, the size of the Type II error β is minimum.

4. The factor $1 - \beta$, the probability of rejecting H_0 when a specific alternative H_1 is true is called the *power of a test*.

22.4 LARGE SAMPLES TESTING

In case of large sampling the test statistic z, say $z = \dfrac{\bar{x} - E(\bar{x})}{S.E.(\bar{x})}$, is approximated to $N(0, 1)$. If z_α is the critical value of the test statistic at level of significance α, then for a *two-tailed test* it is given by $P(|z| > z_\alpha) = \alpha$, that is, z_α is the value so that the total area of the critical region on both tails is α and since the standard normal probability curve is symmetrical about its mean $z = 0$, thus

$$P(z > z_\alpha) = P(z < -z_\alpha) = \alpha/2,$$

as shown in Fig. 22.2a.

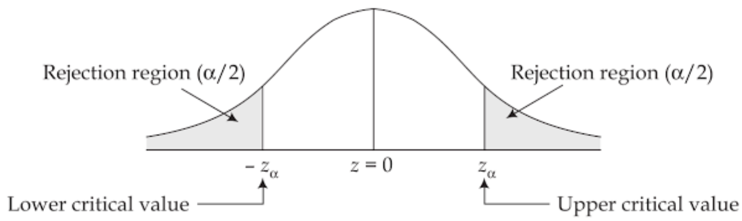


Fig. 22.2a

In case of left-tailed test or right-tailed test the total area to the left of $-z_\alpha$, or to the right of z_α, as the case is, is α. That is, for the left-tailed test $P(z < -z_\alpha) = \alpha$ and for the right-tailed test $P(z > z_\alpha) = \alpha$, as shown in Figs. 22.2b and 22.2c, respectively.

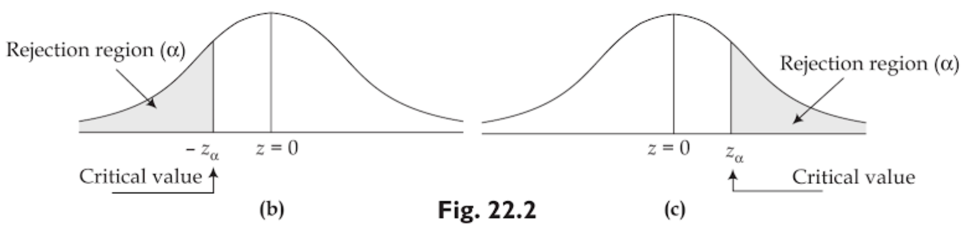


Fig. 22.2

Remarks:

1. In case of large sample testing, the critical value z_α of z for two-tailed test at level of significance α is numerically same as the critical value for a single-tailed test at level of significance $\alpha/2$, but this does not hold good when the sample size is small.

2. When test statistic is approximated to standard normal distribution, for two-tailed test critical values at 1% and 5% level of significance are 2.58 and 1.96 respectively. For left-tailed tests, these values are –2.33 and –1.645 respectively and, for right-tailed tests, the values are 2.33 and 1.645 respectively, see Table I (p. 527)

3. In case the sample size n is small, say <30, and sampled population is not normal then distribution of the test-statistic cannot be approximated to normal and thus these critical values don't hold good. In such cases the values based on the exact sampling distribution of the test statistic are used e.g. t, χ^2, F, etc.

Procedure for testing: We set up the null hypothesis H_0 and compute the test statistic z under the assumption that H_0 is true. If $|z| > 3$, H_0 is rejected outright. In case $|z| \le 3$, we test its significance at a specified level usually at 5% or 1% level of significance.

For a two-tailed test, if $|z| > 1.96$, H_0 is rejected at 5% level of significance and if $|z| > 2.58$ also, it is rejected even at 1% level of significance also. In case $|z| < 2.58$, H_0 may be accepted at 1% level of significance, and if $|z| < 1.96$, then H_0 may be accepted at 5% level of significance also.

Similarly, for large n the test statistic z for proportion of successes $\hat{p}$ given by

$$z = \frac{\hat{p} - p}{\sqrt{pq/n}} \qquad \text{...(22.12)}$$

is a standard normal variate, that is, $z \sim N(0, 1)$.

Further, the limits

$$x = np \pm 1.96\sqrt{npq} \qquad \text{...(22.13)}$$

are called *the 95% confidence limits,* and the limits

$$x = np \pm 2.58\sqrt{npq} \qquad \text{...(22.14)}$$

are called *the 99% confidence limits* for the number of successes x.

Similarly we can write the 95% and 99% confidence limits for the proportion of successes.

Example 22.4: A dice is thrown 9000 times and a throw of 3 or 4 is observed 3240 times. Can the dice be regarded as unbiased? Also find the limits between which the probability of a throw of 3 or 4 is most likely to lie.

Solution: Let the null hypothesis H_0 be that dice is unbiased. Under H_0, if p is the probability of getting a throw of 3 or 4, then we test

$$H_0: p = 1/3, \text{ against the alternative } H_1: p \neq 1/3.$$

Here $n = 9000$, $x = 3240$, $np = 3000$, $q = 2/3$. Thus $\sqrt{npq} = \sqrt{9000 \times 1/3 \times 2/3} = 44.72$

Under H_0 the test statistic z, given by

$$|z| = \frac{|x - np|}{\sqrt{npq}} = \frac{240}{44.72} = 5.37 > 3$$

is highly significant and hence the hypothesis H_0 is rejected and we regard that dice is almost certainly biased.

Since, the dice is not unbiased the most likely limits in which the probability of a throw of 3 or 4 lie are given by

$$\hat{p} \pm 3\sqrt{\frac{\hat{p}\hat{q}}{n}}, \text{ where } \quad \hat{p} = \frac{x}{n} = \frac{3240}{9000} = 0.36, \quad \text{and} \quad \hat{q} = 0.74$$

Hence the limits are, $\quad 0.36 \pm 3\sqrt{\dfrac{(0.36)(0.64)}{9000}} = 0.345$ and 0.375.

Example 22.5: During testing in a sample of 300 chips, 10 have been found to be defective. Can the manufacturer's claim that 2% of the chips are defective may be accepted?

Solution: Let the null hypothesis H_0 be that 2% of the chips are defective, thus we test

$$H_0 : p = .02 \text{ against } H_1 : p \neq .02.$$

Example 22.8: In a random sample of 525 families owning television set in the region of New Delhi it is found that 370 subscribe to Star Plus. Find a 95% confidence interval for the actual proportion of such families in New Delhi which subscribe to Star Plus.

Solution: We have, $\hat{p} = x/n = 370/525 = 0.705.$

Therefore, the 95% confidence limits for the actual proportion p are

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}\hat{q}}{n}} = 0.705 \pm 1.96\sqrt{\frac{(0.705)(0.295)}{525}}\text{, that is, between 0.666 to 0.744.}$$

Hence, 95% confidence interval for the actual proportion p is $0.666 < p < 0.744$.

22.5.2 Test for Difference Between Two Proportions

Suppose we want to compare two distinct populations with respect to the prevalence of a specific attribute. For example, we may be interested in comparing the prevalence of lung cancer among smokers (population I) and non-smokers (population II). Let x_1, x_2 be the number of persons with this attribute in random samples of size n_1 and n_2 selected from population I and population II, respectively. Then sample proportions are

$$\hat{p}_1 = x_1/n \quad \text{and} \quad \hat{p}_2 = x_2/n.$$

If p_1 and p_2 are proportion for the two populations, then

$$E(\hat{p}_1) = p_1, \quad E(\hat{p}_2) = p_2, \quad \text{and} \quad \text{var}(\hat{p}_1) = \frac{p_1 q_1}{n_1}, \quad \text{var}(\hat{p}_2) = \frac{p_2 q_2}{n_2}.$$

Since for large samples, $\hat{p}_1$ and $\hat{p}_2$ are each approximately normally distributed with means p_1 and p_2 and variances p_1q_1/n_1 and p_2q_2/n_2 respectively also the samples being independent, thus $\hat{p}_1 - \hat{p}_2$ is also normally distributed with mean

$$E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2,$$

and,
$$\text{var}(\hat{p}_1 - \hat{p}_2) = \text{var}(\hat{p}_1) + \text{var}(\hat{p}_2) = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}.$$

Thus,
$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} \sim N(0, 1) \qquad \text{....(22.15)}$$

Let H_0: There is no difference between the population proportions, that is, $p_1 = p_2 = p$, say.

Under H_0, $E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2 = 0,$

and,
$$\text{var}(\hat{p}_1 - \hat{p}_2) = \frac{pq}{n_1} + \frac{pq}{n_1} = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right).$$

Hence under H_0, the test statistic (22.15) becomes

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad ...(22.16)$$

Normally p is unknown so we use $\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$, an unbiased estimate of p, in place of p. Thus, in this case the required test statistic z under H_0 is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0.\ 1). \qquad ...(22.17)$$

Example 22.9: Suppose that a method A results in 20 unacceptable transistors out of 100 produced, whereas another method B results in 12 unacceptable transistors out of 100 produced. Can we conclude at 5% level that the two methods are equivalents.?

Solution: Let p_1 and p_2 be the true proportions of unacceptable and let the null hypothesis be that method I and method II are equivalent, that is, we test

$$H_0: p_1 = p_2 \text{ against the alternative } H_1: p_1 \neq p_2$$

We have, $\hat{p}_1 = \frac{20}{100} = 0.20, \quad \hat{p}_2 = \frac{12}{100} = 0.12$

$$\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{20 + 12}{100 + 100} = 0.16.$$

Under H_0 the test statistic z given by

$$|z| = \frac{|\hat{p}_1 - \hat{p}_2|}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{|0.20 - 0.12|}{\sqrt{(0.16)(0.84)\left(\frac{1}{100} + \frac{1}{100}\right)}} = \frac{0.08}{0.052} = 1.538 < 1.96.$$

Thus it is not significant at 5% level and hence the null hypothesis may be accepted.

Example 22.10: An alternate manufacturing mechanism is being tested. Samples are taken using both the existing and the alternate mechanism so as to determine if the alternate mechanism results in an improvement. If 50 of 1000 items from the existing mechanism and 60 of 1500 items from the alternate mechanism were found to be defective, find a 90% confidence interval for the true difference of defectives between the two mechanisms. Can you conclude that alternate mechanism decreases the proportion of defectives significantly?

Solution: Let p_1 and p_2 be the true proportions of defectives in the existing and alternate mechanism respectively.

We have, $\hat{p}_1 = 50/1000 = 0.05$ and $\hat{p}_2 = 60/1500 = 0.04$ and thus

Thus, $$|z| = \frac{|(0.04)-(0.1)|}{\sqrt{(0.217)(0.783)\left(\frac{1}{200}+\frac{1}{100}\right)}} = \frac{0.06}{0.0505} = 1.18.$$

Since $|z| = 1.18 < 1.96$, it is not significant at 5% level of significance and hence null hypothesis may be accepted and thus company's claim may be considered to be valid one.

Example 22.12: The percentage of officials in two big PSU's with computer knowledge is 30 and 25, respectively. Is this difference likely to be hidden in samples of 1000 and 800 officials, respectively from the two PSU's?

Solution: Let p_1 and p_2 be the true proportions of the officials in the two PSU's and let the null hypothesis be that difference is likely to be hidden, that is,

$H_0 : \hat{p}_1 = \hat{p}_2$ against the alternative H_1: $\hat{p}_1 \neq \hat{p}_2$

We have, $n_1 = 1000$, $n_2 = 800$, $p_1 = 0.30$, $p_2 = 0.25$.

Under H_0, the test statistic z given by

$$z = \frac{|p_1 - p_2|}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = \frac{0.30 - 0.25}{\sqrt{\frac{(0.3)(0.7)}{1000} + \frac{(0.25)(0.75)}{800}}} = \frac{0.05}{0.021} = 2.37 > 1.96,$$

is significant at 5% level and hence the hypothesis is rejected and thus the difference is likely to be revealed in the samples drawn at 5% level of significance.

We observe that, since $|z| = 2.37 < 2.58$ is not significant at 1% level and so that hypothesis may be accepted at 1% and hence the samples are unlikely to reveal the difference at 1% level of significance.

22.6 SAMPLING OF VARIABLES. TESTS FOR SINGLE MEAN AND DIFFERENCE BETWEEN TWO MEANS

In this section, we discuss the sampling of the values of a variable such as height, weight, marks obtained in a test, weekly wages, etc. Each unit of the population provides a specific value (measurement) of the variable under study and the aggregate of these values forms the frequency distribution of the population. From this population, (that is, the aggregate of the values), a random sample of size n is selected to estimate and draw the conclusions about the population parameters, generally unknown.

22.6.1 Test for Single Mean

For large sample size n, sample mean is distributed normally with its mean as sampled population mean μ and S.E. as $\sigma/\sqrt{n}$, where σ is the S.D. of the sampled population. Thus under the null hypothesis H_0, that the sample has been drawn from a population with mean μ and S.D. σ, the test statistic z given by

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1), \qquad \text{...(22.18)}$$

is a standard normal variate with mean zero and S.E. one, when n is large.

If $|z| < 1.96$ the deviation is not significant at 5% and the hypothesis is accepted, otherwise, it is rejected.

The limits $\bar{x} \pm 1.96\left(\sigma/\sqrt{n}\right)$ are the *95% confidence limits for the population mean* μ; and

$$\bar{x} - 1.96\left(\sigma/\sqrt{n}\right) < \mu < \bar{x} + 1.96\left(\sigma/\sqrt{n}\right) \qquad \text{...(22.19)}$$

is the *95% confidence interval.*

Similarly, $\bar{x} \pm 2.58\left(\sigma/\sqrt{n}\right)$ are the *99% confidence limits for the population mean* μ.

Remark: In case population S.D. σ is unknown we use s, the sample S.E. as its estimate, for $s^2 = \frac{n-1}{n}S^2 = \left(1 - \frac{1}{n}\right)S^2$ and thus for large n, $s^2 \to S^2$ and further $E(S^2) = \sigma^2$ justifies s^2 as an estimate of σ^2 in case latter is unknown.

Example 22.13: Sugar is packed in bags by an automatic machine with mean contents of bags as 1.000 kg. A random sample of 36 bags is selected and mean mass has been found to be 1.003 kg. If a S.D. of 0.01 kg, is acceptable on all the bags being packed, determine on the basis of sample test whether the machine requires adjustment.

Solution: Let the null hypothesis H_0 be that the machine does not require any adjustment, that is,

$$H_0: \mu = 1.000 \text{ kg. against } H_1: \mu \neq 1.000 \text{ kg.}$$

Under H_0, the statistic z given by

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{1.003 - 1.000}{.01/\sqrt{36}} = 1.8 < 1.96$$

Thus, it is not significant at 5% level and hence H_0 may be accepted, that is, machine does not require any adjustment.

Example 22.14: The daily collection of milk at a plant has averaged 850 kilolitres for the last several years. An observer wants to know whether the average has changed in recent months. He randomly selects 40 days from the database and finds the average collection as $\bar{x} = 840$ kilolitres with a S.D. $s = 18$ kilolitres. Test the appropriate hypothesis at $\alpha = 0.05$.

Solution: We test the null hypothesis H_0: $\mu = 850$, against H_1: $\mu \neq 850$.

Under H_0 the test statistic z, given by

$$|z| = \frac{|\bar{x} - \mu|}{s/\sqrt{n}} = \frac{|840 - 850|}{18/\sqrt{40}} = 3.51 > 1.96.$$

Thus it is significant at 5% (even it is significant at 1% level) and hence hypothesis is rejected, that is, daily average collection of milk has changed.

Example 22.15: If e is the permissible error for estimating the population parameter μ, then prove that the minimum sample size n required for estimating μ with 95% confidence is given by $n = (1.96\sigma/e)^2$, where σ^2 is the population variance.

Solution: For large n, the test statistic z is given by

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Thus, $$P\left\{|x - \mu| \le 1.96\frac{\sigma}{\sqrt{n}}\right\} = 0.95$$

We need n such that $P\{|x - \mu| < e\} > 0.95$.

Comparing these two, we obtain

$$\text{min. } e = \frac{1.96\sigma}{\sqrt{n}}, \text{ thus } \frac{1.96\sigma}{\sqrt{n}} \le e, \text{ which gives, } n \ge \left(\frac{1.96\sigma}{e}\right)^2.$$

Hence, $$\text{min. } n = \left(\frac{1.96\sigma}{e}\right)^2.$$

Remark. For 99% confidence, min. $n = \left(\frac{2.58\sigma}{e}\right)^2$.

Example 22.16: The average zinc concentration recovered from a sample of zinc measurements in 40 different locations is found to be 2.54 gm per millitre. Find the 95% confidence intervals for the mean zinc concentration in the river assuming the population S.D. to be 0.32 gm. Find the minimum sample size required at 95% confidence if the permissible error is 0.05 gm.

Solution: We have, $\bar{x} = 2.54$, $n = 40$, $\sigma = 0.32$.

For large n the statistic z is given by

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1),$$

where μ is the population mean.

Thus, 95% confidence interval for μ is

$$\bar{x} - (1.96)\,\sigma/\sqrt{n} < \mu < \bar{x} + 1.96\,\sigma/\sqrt{n}$$

or, $$2.54 - (1.96)\frac{0.32}{\sqrt{40}} < \mu < 2.54 + 1.96\frac{0.32}{\sqrt{40}},$$

which simplifies to $2.44 < \mu < 2.63$.

The minimum sample size n required at 95% confidence is

$$n = \left(\frac{1.96\sigma}{e}\right)^2 = \left(\frac{(1.96)(0.32)}{0.05}\right)^2 = 157.35 \simeq 158.$$

Example 22.17: The average monthly earnings for women in executive positions is Rs. 33,500. A random sample of n = 40 men in the executive positions showed average monthly earning $\bar{x}$ = Rs. 36,250, with S.E. s = Rs. 5100. Do men in the same position have average monthly earnings higher than those for women?

Solution: We test H_0: μ = 33,500 against the right-tailed alternate $H_1 : \mu > 33{,}500$.

Under H_0, the test statistic z, is given by

$$z = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim N(0, 1).$$

We have,
$$z = \frac{36250 - 33500}{5100/\sqrt{40}} = 3.41 > 1.645,$$

the value of z from Table I at α = .05, for right-tailed test.

Hence calculated z is significant and thus hypothesis is rejected. Thus men in the same positions have higher salary then their females counterparts.

22.6.2 Test for Difference Between Two Means

In many situations we are concerned with the comparison of two population means. For example, our problem of concern may be the comparison of the lead levels in drinking water in two different sections of a city.

Let $\bar{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1 and variance σ_1^2, and let $\bar{x}_2$ be the mean of an independent sample of size n_2 from a popution with mean μ_2 and variance σ_2^2. Since the two samples are independent for large values of n_1 and n_2, $\bar{x}_1 - \bar{x}_2$ can be approximated to a normal variate with mean and variance respectively as

$$E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu_1 - \mu_2;$$

and,
$$\text{var}(\bar{x}_1 - \bar{x}_2) = \text{var}(\bar{x}_1) + \text{var}(\bar{x}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

Thus the statistic z given by

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \qquad \text{...(22.20)}$$

is a standard normal variate with mean zero and S.E.one, that is, $z \sim N(0, 1)$.

Under the null hypothesis H_0: $\mu_1 = \mu_2$, that is, there is no significant difference between the two population means, the statistic (22.20) becomes

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1). \qquad \text{...(22.21)}$$

and hence can be tested accordingly.

Solution: We have, $n_1 = 500, \quad \bar{x}_1 = 28.57, \quad s_1 = 1.25$

$n_2 = 400, \quad \bar{x}_2 = 29.62, \quad s_2 = 1.42.$

Let the null hypothesis be that the samples have been drawn from the same population with S.D. σ, that is, we test

$H_0 : \mu_1 = \mu_2$ with S.D. σ against the alternative $H_1 : \mu_1 \neq \mu_2$.

Under H_0, the test statistic z is given by

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0, 1)$$

Since σ^2 is not given, we use

$$\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2} = \frac{500(1.25)^2 + 400(1.42)^2}{500 + 400} = 1.764$$

as an estimate to σ^2.

Thus, we have $$z = \frac{28.57 - 29.62}{1.328\sqrt{\frac{1}{500} + \frac{1}{400}}} = \frac{-1.05}{0.0891} = -11.78.$$

Since, $|z| = 11.78 > 3$, the value is highly significant, and hence the null hypothesis is rejected. Samples cannot be assumed to be drawn from the population.

EXERCISE 22.2

1. In a survey of 100 adults over 40 years old, a total of 15 people were found to be participating in a fitness activity at least twice a week. Test the hypothesis at $\alpha = .05$ that the participation rate for adult over 40 years of age is not less than the 20% figure.
2. A sleep inducing tablet when administered to 50 insomniacs was found to be effective on 37 patients. Test the hypothesis at $\alpha = 0.05$ that tablet was effective in at least 80% cases.
3. A random sample of 500 fuses was taken from a large consignment and out of these 60 were found to be defective. Obtain the 98% confidence limits for the percentage of defective fuses in the consignment.
4. In a locality containing 18000 families, a sample of 840 families was selected at random. Of these 840 families, 206 families were found to have a daily earning of Rs. 250 or less. Estimate the almost certain limits within which such families are likely to lie.
5. In a city A, 20% of a random sample of 900 Sr. Sec. school boys had computer knowledge, while in another city B, 18.5% of a random sample of 1600 Sr. Sec. school boys had computer knowledge. Test the hypothesis that there is no difference in proportions of boys with computer knowledge among Sr. Sec. students in the two cities.
6. In two large cities out of the houses with cable connections 30% and 25% respectively have Discovery channel. Is this difference likely to be hidden in samples of 1200 and 900 such houses from the two cities?

7. A study shows that 16 of 200 tractors produced on one assembly line required extensive adjustment before they could be shipped, while the same was true for 14 of 400 tractors produced on another assembly line. At the 0.01 level of significance does this support the claim that the second production line does superior work?

8. An airline claims that only 6% of all lost luggage is never found. If in a random sample, 17 of 200 pieces of lost luggage are not found, test the null hypothesis $p = 0.06$ against the alternative hypothesis $p > 0.06$ at the 0.05 level of significance.

9. To compare two different types of paints, eighteen specimens are painted using type *A* and the drying time in hours is recorded on each. Then the same is done with type *B*. If $\bar{x}_A$ and $\bar{x}_B$ are the mean drying times in hours for types *A* and *B* respectively, find $p[\bar{x}_A - \bar{x}_B > 1.0]$ under the assumption that population mean drying times for two types *A* and *B* are same with S.D. of 1.0.

10. An insurance agent claims that the average age of policy-holders who insure through him is less than the average for all agents which is 30.5 years. A random sample of 100 policy-holders insured through him gave the following age distribution.

Age as on last birthday :	16-20	21-25	26-30	31-35	36-40
No. of persons :	12	22	20	30	16

Test his claim at the 5% level on the basis of the data obtained.

11. A survey is proposed to be conducted to estimate the monthly income of the alumni of a technical institution. How large should the sample be taken in order to estimate the annual earning within plus and minus Rs. 10,000 at 95% confidence level, assuming the S.D. of the annual earnings of the entire alumni is known to be Rs. 30,000?

12. A taxi company is to decide whether to purchase brand *A* or brand *B* tires for its fleet of taxis. To estimate the difference in the two brands, an experiment is conducted using 30 tires of each brand. The tires are run until they wear out. The results are

$$\bar{x}_A = 36{,}300 \text{ kilometre} \qquad \bar{x}_B = 38{,}100 \text{ kilometre}$$
$$s_A = 5{,}000 \text{ kilometre} \qquad s_B = 6{,}100 \text{ kilometre}$$

Compute a 95% confidence interval for $\mu_B - \mu_A$ assuming the populations to be normal.
What do you conclude from confidence interval obtained?

13. A random sample of 100 pieces was immersed in a bath for 24 hrs yielding an average of 12.2 millimetres of metal removed and a sample S.D. of 1.1 millimetre. A second sample of 200 pieces was exposed to some treatment, followed by the 24 hours immersion in the bath, resulting in an average removal of 9.1 millimetre with a sample S.D. of 0.9 millimetre. Compute a 98% confidence interval for the difference between the population means. Does the treatment appear to reduce the mean amount of metal removed ?

14. The mean breaking strength of cables supplied by a manufactures is 1800 with a S.D. 100. By a new technique in the manufacturing process it is claimed that the breaking strength has increased. In order to test this claim a sample of 50 cables is tested. It is found that the mean breaking strength is 1850. Can we support the claim at $\alpha = 0.01$?

15. The electric light tubes of type *A* have a lifetime of 1400 hrs with a S.D. of 200 hrs, while of type *B* have mean lifetime of 1200 hrs with a S.D. of 100 hrs. If random samples of 125 tubes of each batch are tested, what is the probability that the type *A* tubes will have a mean time

which is at least, (a) 160 hrs. more than the type B tubes, and (b) 250 hrs. more than the type B tubes?

22.7 SMALL SAMPLES TESTING. STUDENT'S t-VARIATE AND ITS APPLICATIONS

So far we have discussed the large samples testing. All these tests were based on central limit theorem to justify the normality of the test statistic derived. In case we are unable to collect a large sample, the test procedures described so far are of no use. Here we study equivalent procedures which can be employed when the sample size is small. However, we shall assume that the population (s) from which the samples are drawn is normal.

22.7.1 Student's t-Variate

When the sampled population is normal, the statistic $z = (\bar{x} - \mu)/(\sigma/\sqrt{n})$ has normal distribution for any sample size n, small or large. In case the population S.D. σ is unknown and the sample size n is small, the statistic $(\bar{x} - \mu)/(S/\sqrt{n})$, where $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$ is no longer distributed as a normal variate and we define this statistic as 't', that is,

$$t = \frac{(\bar{x} - \mu)}{S/\sqrt{n}}. \qquad \text{...(22.24)}$$

The statistic t defined as in (22.24) and its distribution, was mathematically derived by W.S. Gosset in 1908. He published his work under the pen name 'Student' and hence the distribution is known as Student's t, distribution. The probability density function of the t-variate (22.24) is given by

$$f(t) = \frac{1}{\sqrt{\nu}\,B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \cdot \frac{1}{\left(1 + \frac{t^2}{\nu}\right)^{(\nu+1)/2}}, \quad -\infty < t < \infty, \qquad \text{...(22.25)}$$

where $\nu = n - 1$.

The probability curves for some specific values of $\nu = n - 1 = 2, 5$ are shown in Fig. 22.3

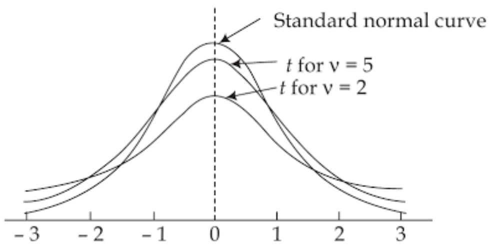


Fig. 22.3

The chief characteristics of the t-statistic are:

1. It is mound shape and symmetric about $t = 0$, but is more flat on the top than the normal curve.

2. From Fig. 22.3 we observe that the t curve does not approach the horizontal axis as fast as standard normal curve approaches, in fact for small n, $p(|t| \geq t_0) \geq p(|z| \geq t_0)$; $z \sim N(0, 1)$.

3. The shape of the t curve depends on the sample size n and when $n \to \infty$ the distribution of t tends to standard normal.

4. Since, $f(t)$ is symmetrical about the line $t = 0$, all the odd order moments of about $t = 0$ are zeros, that is, $\mu'_{2r+1} = 0$. In particular μ, the mean is zero. Hence the central moments, that is, moments about mean coincide with moment about zero and so $\mu_{2r+1} = 0$.

Also we can show that the even order moments are given by

$$\mu_{2r} = \nu^r \frac{(2r-1)(2r-3)\ldots 3.1}{(\nu-2)(\nu-4)\ldots(\nu-2r)}, \quad \nu > 2r. \qquad \ldots(22.26)$$

In particular

$$\mu_2 = \frac{\nu}{\nu-2}, \quad \nu > 2 \text{ and } \mu_4 = \frac{3\nu^2}{(\nu-2)(\nu-4)}, \quad \nu > 4.$$

Hence,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = 0, \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} = 3\left(\frac{\nu-2}{\nu-4}\right), \quad \nu > 4. \qquad \ldots(22.27)$$

As $\nu \to \infty$, $\beta_2 \to 3$, that is, the curve tends to normal.

5. The distribution of the random variable t as defined in (22.24) is called the t distribution with $(n-1)$ *degrees of freedom* (df) and the variate is represented as t_{n-1} or t_ν, where $\nu = n - 1$.

Degrees of freedom $\nu = n - 1$ is the quantity by which $\sum_{i=1}^{n} (x_i - \overline{x})^2$ is divided in order to obtain an unbiased estimate of σ^2 and refers to the amount of information available in the data used for estimating σ^2. For each possible value of df $\nu = n - 1$, there is a different t distribution and as ν approaches large, the t-variate tends to z-variate, the standard normal variate.

In particular, for $\nu = 1$, that is, for $n = 2$, (22.25) gives

$$f(t) = \frac{1}{\beta\left(\frac{1}{2}, \frac{1}{2}\right)} \frac{1}{(1+t^2)}; \quad -\infty < t < \infty$$

$$= \frac{1}{\pi(1+t^2)}; \quad -\infty < t < \infty, \qquad \ldots(22.28)$$

the p.d.f. of a standard *Cauchy variate.*

The t-variate has vital applications in statistical inference. Although distribution of t is asymptotically normal for large n, but for small n it considerably differs from normal. The discovery of t variate have led to many important contributions in the development of small sample theory.

Significant *values* of *t*: The significant values of t at level of significance α and degrees of freedom ν for two-tailed test, denoted by $t_{\nu[\alpha]}$, are given by

$$P[\,|t| > t_{\nu[\alpha]}] = \alpha \qquad ...(22.29)$$

or,
$$P[\,|t| < t_{\nu[\alpha]}] = 1 - \alpha. \qquad ...(22.30)$$

The Table II (p. 528) gives the values $t_{\nu[\alpha]}$ of t-distribution (two-tail areas) for different values of α and ν, as shown in Fig. 22.4.

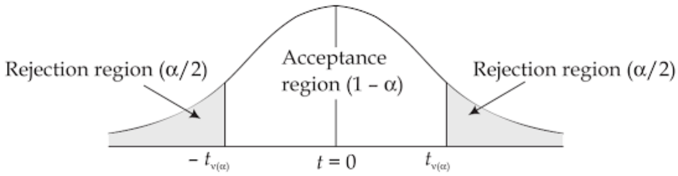


Fig. 22.4

Since the distribution is symmetrical about $t = 0$, (22.29) gives

$$P[t > t_{\nu[\alpha]}] + P[t < -t_{\nu[\alpha]}] = \alpha$$

which implies, $2P[t > t_{\nu[\alpha]}] = \alpha$, or $P[t > t_{\nu[\alpha]}] = \dfrac{\alpha}{2}$, or $P[t > t_{\nu[2\alpha]}] = \alpha$.

Thus, the significant values of t at level of significance α for single-tailed tests (left or right) are those of two-tailed test at level of significance 2α.

For example, from Table II

$$t_{10[0.05]} \text{ for single-tail test} = t_{10[.10]} \text{ for two-tailed test} = 1.81$$

We should note that the significant values of t lead us to reliable inferences about the sampled population if the sample drawn meets the following requirements.

1. The sample must be randomly selected.
2. The sampled population must be normally distributed.

22.7.2 Test for Single Mean

Given a random sample $x_1, x_2, \ldots x_n$ from a normal population. We need to test the hypothesis that the mean of the sampled population is μ. Under the null hypothesis H_0, the statistic

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}, \qquad ...(22.31)$$

where $\bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$ and $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$ follows t-distribution with $\nu = n - 1$ df.

We compare the calculated value of t with the tabulated value at the desired level of significance α. If calculated $|t|$ is greater than the tabulated t, null hypothesis is rejected and if calculated $|t|$ is less than the tabulated t, the null hypothesis is accepted at the level of significance α.

If $t_{\nu[0.05]}$ is the tabulated value of t for $\nu = n - 1$ *df* at 5% level of significance, then *95% confidence limits for the population mean* μ are given by $\bar{x} \pm t_{\nu[0.05]}\left(S/\sqrt{n}\right)$. Similarly, *99% confidence limits for the population mean* μ are given by $\bar{x} \pm t_{\nu[0.01]}\left(S/\sqrt{n}\right)$.

Example 22.22: Ten individuals were chosen at random from a normal population and their heights were found to be in inches as 63, 63, 66, 67, 68, 69, 70, 70, 71 and 71. Test the hypothesis that the mean height of the population is 66 inches. Also find the 95% confidence limits for the true population mean μ.

Solution: We have $\bar{x} = 67.8$ inches

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{(\Sigma\, x_i)^2}{n}\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} d_i^2 - \frac{(\Sigma d_i)^2}{n}\right], \quad d_i = x_i - 68$$

$$= \frac{1}{9}\left[82 - \frac{4}{10}\right] = 9.067$$

or, $S = 3.011$ inches.

We test the null hypotheses H_0: $\mu = 66$ against two-tailed alternative H_1: $\mu \neq 66$.

Under H_0, $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}} \sim t_\nu$; the t-variate with $\nu = n - 1 = 9$ df.

We have, $t = \dfrac{67.8 - 66}{3.011/\sqrt{10}} = 1.89.$

From Table II, $t_{9[.05]} = 2.26$. Since t calculated is less than t tabulated hence the null hypotheses H_0 may be accepted at 5% level of significance.

Also the 95% confidence limits for the population mean μ are:

$$\mu = \bar{x} \pm t_{\nu[0.05]}\left(S/\sqrt{n}\right)$$

$$= 67.8 \pm (2.26)\left(3.011/\sqrt{10}\right) = 67.8 \pm 2.53.$$

Example 22.23: The mean weekly sales of TVs of a particular brand in company's showrooms was 14.6 TV per showroom. After announcing a few incentives the mean weekly sales in 22 stores for a typical week increased to 15.4 with S.D. of 1.7. Were the incentives announced effective in boosting the sale?

Solution: We have, $n = 22$, $\bar{x} = 15.4$, $s = 1.7$. Let the null hypothesis H_0 be that incentive announced were not effective. Thus we test

$H_0 : \mu = 14.6$ against the right tailed alternative $H_1 : \mu > 14.6$.

Under H_0,
$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \sim t_\nu.$$

We have,
$$t = \frac{15.4 - 14.6}{1.7/\sqrt{21}} = \frac{\sqrt{21}(0.8)}{1.7} = 2.16.$$

From Table II, $t_{21[0.05]}$ for single-tail test = $t_{21[0.10]}$ for two-tailed test = 1.72.

Since calculated value of t is greater than the tabulated value so hypothesis H_0 is rejected at 5% level of significance, that is, incentives announced were effective.

Example 22.24: The specifications for a certain kind of ribbon call for a mean breaking strength of 180 pounds. If five randomly selected pieces of the ribbon have a mean breaking strength of 169.5 pounds with a S.D. of 5.7 pound, test the null hypothesis $\mu = 180$ against the alternative hypothesis $\mu < 180$ at 5% level of significance.

Solution: We have, $n = 5$, $\bar{x} = 169.5$, $s = 5.7$. We test the null hypothesis

$H_0 : \mu = 180$ against the left-tailed alternative $H_1 : \mu < 180$.

Under H_0,
$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \sim t_\nu .$$

We have
$$|t| = \frac{|169.4 - 180|}{5.7/\sqrt{n-1}} = |-3.72| = 3.72.$$

From Table II, $t_{4[0.05]}$ for single-tailed test = $t_{4[0.1]}$ for double-tailed test = 2.13.

Since t calculated is greater than the t tabulated, the null hypothesis H_0 is rejected at 5% level of significance.

22.7.3 Test for Difference Between Two Means

We shall consider tests of significance for following different cases:

Case I: Given two independent random samples $x_1, x_2, ..., x_{n_1}$ and $y_1, y_2, ..., y_{n_2}$ with means $\bar{x}$ and $\bar{y}$ and standard deviations s_x and s_y from normal populations with means μ_x and μ_y and with the same variances. We need to test the hypothesis that the population means are the same, that is, samples have been drawn from the same normal population.

$$S_1^2 = \frac{1}{4}\sum_{i=1}^{5}(x_i - \bar{x})^2 = \frac{63{,}000}{4} = 15{,}750, \quad S_2^2 = \frac{1}{5}\sum_{i=1}^{6}(y_i - \bar{y})^2 = \frac{54{,}600}{5} = 10{,}920$$

Under H_0, $$t = \frac{\bar{x} - \bar{y}}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_\nu;\ \nu = n_1 + n_2 - 2$$

Also, $$S^2 = \frac{\sum_{i=1}^{5}(x_i - \bar{x})^2 + \sum_{i=1}^{6}(y_i - \bar{y})^2}{n_1 + n_2 - 2} = \frac{63000}{9} + \frac{54600}{9} = 13066.67$$

Thus $$t = \frac{8230 - 7940}{114.31\sqrt{\frac{1}{5} + \frac{1}{6}}} = 4.19.$$

From Table II, $t_{9[0.05]} = 2.26$. Since t calculated is greater than t tabulated, hence hypothesis is rejected at 5% level of significance.

Example 22.26: Samples of two types of electric light bulbs were tested for length of life and following data were obtained

	Type I	Type II
Sample sizes	$n_1 = 8$	$n_2 = 7$
Sample means	$\bar{x}_1 = 1234$ hrs	$\bar{x}_2 = 1036$ hrs
Sample S.D.'s	$s_1 = 36$ hrs	$s_2 = 40$ hrs.

Does the data support the hypothesis that Type I is superior to Type II regarding length of life ?

Solution: Let μ_1, μ_2 be the population means, we test the null hypothesis,

$H_0 : \mu_1 = \mu_2$ against right-tailed alternative $H_1 : \mu_1 > \mu_2$

Under H_0, $$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_\nu;\ \nu = n_1 + n_2 - 2$$

Here $$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{1}{13}[8(36)^2 + 7(40)^2] = \frac{1}{13}[10368 + 11200] = 1659.08.$$

Thus $$t = \frac{1234 - 1036}{40.73\left(\sqrt{\frac{1}{8} + \frac{1}{7}}\right)} = \frac{198}{21.08} = 9.39.$$

From Table II, $t_{13[0.05]}$ for single-tailed test = $t_{13[0.10]}$ for two-tailed test = 1.77.

Since t calculated is greater than t tabulated, the hypothesis H_0 is rejected at 5% level of significance.

Example 22.27: The yields of two types Type I and Type II of grains in pounds per acre in 6 replications are given below. Give your comments on the difference in the mean yields.

Replication	*Type I*	*Type II*
1	205	248
2	246	263
3	230	282
4	300	308
5	304	300
6	238	220

Solution: Let the null hypothesis H_0 be that there is no difference in the mean yields of Type I and Type II. We test the null hypothesis

$H_0 : \mu_1 = \mu_2$ against the two-tailed alternative $H_1 : \mu_1 \neq \mu_2$.

Under H_0, the test statistic t in case of paired observation is,

$$t = \frac{\bar{d}}{S/\sqrt{n}} \sim t_\nu; \quad \nu = n - 1$$

We have, $$\bar{d} = \frac{1}{6}\sum_{i=1}^{6} (x_i - y_i) = \frac{-43 - 17 - 52 - 8 + 4 + 18}{6} = -16.3$$

and, $$S^2 = \frac{1}{5}\sum_{i=1}^{6} (d_i - \bar{d})^2$$

$$= \frac{(-26.7)^2 + (0.49)^2 + (-35.7)^2 + (8.3)^2 + (20.3)^2 + (34.3)^2}{5} = 729.07.$$

Thus, $$|t| = \frac{|\bar{d}|}{S/\sqrt{n}} = \frac{|(-16.3)\sqrt{6}|}{27} = 1.48.$$

From Table II, $t_{5[0.05]} = 2.57$. Since t calculated is less than t tabulated hypothesis may be accepted at 5% level of significance.

Case III: Given two random samples $x_1, x_2, ..., x_{n_1}$ and $y_1, y_2, ..., y_{n_2}$ from normal populations with the same variance we need to test the hypothesis that the population means are μ_x and μ_y, respectively.

In this case under H_0 the test statistic t given by

$$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad ...(22.34)$$

follows t distribution with $\nu = (n_1 + n_2 - 2)$ degrees of freedom and thus we test the hypothesis accordingly.

Example 22.28: To test the claim that the resistance of electric wire can be reduced by at least 0.05 ohm by alloying, 25 measurements obtained for each alloyed wire and standard wire produced the following results:

		Mean	S.D.
Alloyed wire (x)	:	0.083 ohm	0.003 ohm
Standard wire (y)	:	0.136 ohm	0.002 ohm

Test the claim at 5% level of significance.

Solution: Let the claim be valid, so we test the null hypothesis

$H_0 : \mu_x - \mu_y \geq 0.05$ against the left-tailed alternative $H_1 : \mu_x - \mu_y < 0.05$.

Under H_0,
$$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t_\nu \quad \nu = n_1 + n_2 - 2$$

Here
$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{(25)(.003)^2 + 25(.002)^2}{48} = .0000067.$$

Therefore,
$$|t| = \frac{|(0.083 - 0.136) - 0.05|}{(0.00260)\sqrt{\left(\dfrac{1}{25} + \dfrac{1}{25}\right)}} = \frac{|-.103|}{.00073} = |-144.09| = 144.09.$$

From Table II, $t_{48[.05]}$ for single-tailed test = $t_{48[.1]}$ for double-tailed test = 1.65.

Since $|t|$ calculated is greater than t tabulated, hypothesis is rejected at 5% level of significance.

22.7.4 Testing the Significance of an Observed Correlation Coefficient

Let r be the observed correlation coefficient in random sample of n observations (x_i, y_i) from a bivariate normal population; we need to test the hypothesis H_0 that the sampled population correlation coefficient ρ is zero.

We can show that under H_0, the test statistic t given by

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \qquad \text{...(22.35)}$$

is a 't' variate with $\nu = (n - 2)$ degrees of freedom and thus we test the hypothesis accordingly.

Example 22.29: A random sample of fifteen paired observations from a bivariate normal population gives a correlation coefficient of –0.5. Does this signify the existence of correlation in the sampled population?

Solution: Let the null hypothesis H_0 be that sampled population is uncorrelated. Thus, we test

$H_0 : \rho = 0$ against the alternate $H_1 : \rho \neq 0$.

8. In a certain experiment to compare two types of animal feed A and B, the following results of increase in weights were observed in two independent samples of animals each of size 8. Test the hypothesis that food B is better than food A.

Increase in weight in lbs

Food A	49	53	51	52	47	50	52	53
Food B	52	55	52	53	50	54	54	53

9. A coefficient of correlation of 0.2 is obtained from a random sample of 625 pairs of observations.

(i) Is this value of r significant?

(ii) Obtain the 95% confidence limits to the correlation coefficient in population; use that when n is large t-variate is distributed like a standard normal variate.

10. Find the least value of r in a sample of 18 pairs of observations from a bivariate population significant at 5% level.

22.8 CHI-SQUARE VARIATE AND TEST FOR POPULATION VARIANCE

While studying the applications of t-variates in tests of significance we have seen that an estimate of the population variance σ^2 is usually required to make inferences about the population mean. Otherwise also in some practical situations the knowledge of the variance of the sampled population may be more important than the population mean. For example, our concern may be to know the precision of a measuring instrument being used, or we may be much concerned about the variation of the water level at different points during flood.

Suppose we want to test if a random sample $x_1, x_2, ..., x_n$ has been drawn from a normal population with a specified variance σ^2. Then under the null hypothesis that the population variance is σ^2, the statistic χ^2 called chi-square variable, defined by

$$\chi^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} = \frac{1}{\sigma^2}\left[\sum_{i=1}^{n} (x_i - \bar{x})^2\right] = \frac{(n-1)S^2}{\sigma^2}, \qquad ...(22.36)$$

where $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$ is an unbiased estimate of σ^2, follows sampling distribution with probability distribution given by

$$\frac{1}{2^{\nu/2}\Gamma(\nu/2)}\cdot\left[\exp\left(-\frac{1}{2}\chi^2\right)\right](\chi^2)^{\frac{\nu}{2}-1} \quad 0 < \chi^2 < \infty. \qquad ...(22.37)$$

The distribution defined by (22.37) is called χ^2 probability distribution with $\nu = n - 1$ degrees of freedom (df). The probability curve for a chi-square distribution is shown in Fig. 22.5. The curve is skewed towards right and its shape varies with the degrees of freedom $\nu = n - 1$. The variate tends to standard normal variate as $n \to \infty$.

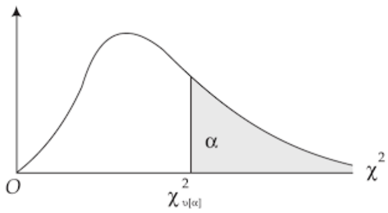


Fig. 22.5

Critical values and test of significance. Let $\chi^2_{\nu[\alpha]}$ denote the value of chi-square variate for ν *df* such that the area to the right of this point is α, that is, $P[\chi^2 > \chi^2_{\nu[\alpha]}] = \alpha$, as shown in Fig. 22.5. The Table III (p. 529) gives the critical values or significant values of $\chi^2_{\nu[\alpha]}$ for the right-tailed test for different degrees of freedom ν and significant level α. We observe that value of $\chi^2_{\nu[\alpha]}$ increases with increase in ν and decrease in α. At a specific level α and *df* ν, the null hypothesis H_0: $\sigma^2 = \sigma_0^2$ is rejected against the alternate hypothesis

(i) $H_1 : \sigma^2 > \sigma_0^2$; if calculated $\chi^2 > \chi^2_{\nu[\alpha]}$, refer Fig. 22.5

(ii) $H_1 : \sigma^2 < \sigma_0^2$; if calculated $\chi^2 < \chi^2_{\nu[1-\alpha]}$, refer Fig. 22.6

(iii) $H_1 : \sigma^2 \neq \sigma_0^2$; if calculated $\chi^2 > \chi^2_{\nu[\alpha/2]}$ or $\chi^2 < \chi^2_{\nu[1-\alpha/2]}$, refer Fig. 22.7

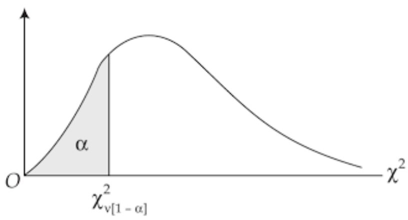


Fig. 22.6

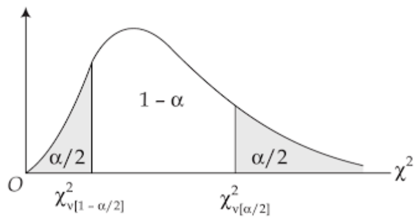


Fig. 22.7

Equal tails are used for the two-tailed χ^2 test as a matter of mathematical convenience only otherwise, chi-square distribution is not symmetric. However, normally in practice right-tailed test is applicable.

Example 22.30: A manufacturer of car batteries claims that the life of his batteries is approximately normally distributed with a S.D. of 0.9 years. If a random sample of 10 of these batteries has a S.D. of 1.1 years, do you think $\sigma > 0.9$ years at $\alpha = 0.05$?

Solution: We test $H_0 : \sigma^2 = 0.81$ against the right-tailed alternative H_1: $\sigma^2 > 0.81$.

We have $n = 10, \quad \sigma^2 = 0.81, \quad s^2 = (1.1)^2 = 1.21$

Under H_0, the test statistic

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{ns^2}{\sigma^2} = \frac{10(1.21)}{0.81} = 14.94$$

follows χ^2 distribution with df $v = n - 1 = 10 - 1 = 9$.

From Table III, $\chi^2_{9[0.5]} = 16.92$. Since χ^2 calculated is less than χ^2 tabulated so value is not significant and hence hypothesis H_0 may be accepted at 5% level of significance.

Example 22.31: Following data give the 11 measurements of the same object on the same instrument:

2.5, 2.3, 2.4, 2.3, 2.5, 2.7, 2.5, 2.6, 2.6, 2.7, 2.5.

At 1% level, test the hypothesis that the variance of the instrument is no more than 0.16.

Solution: We test the null hypothesis $H_0 : \sigma^2 = 0.16$ against alternative $H_1 : \sigma^2 > 0.16$.

For the given data $\quad \bar{x} = \dfrac{27.6}{11} = 2.51, \quad \Sigma (x - \bar{x})^2 = 0.1891.$

Under H_0, the test statistic χ^2, given by

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{\Sigma (x - \bar{x})^2}{\sigma^2} = \frac{0.1891}{0.16} = 1.182$$

follows χ^2 distribution with degrees of freedom $\nu = n - 1 = 11 - 1 = 10$.

From Table III, $\chi^2_{10[.01]} = 23.2$, and since the χ^2 calculated is less than the χ^2 tabulated, so hypothesis may be accepted at 1% level of significance.

22.9 F-VARIATE AND TEST FOR THE EQUALITY OF TWO POPULATION VARIANCES

Sometimes we need to compare two population variances. For example, we may be interested to compare the precisions of the two measuring instruments, or we may by interested in the stability of measurement on a manufactured product from two assembly lines. Suppose we want to test whether the two independent samples $x_1, x_2, ..., x_{n_1}$ and $y_1, y_2, ..., y_{n_2}$ have been drawn from the normal populations with the same variance σ^2. Under the null hypothesis that the population variances σ_x^2 and σ_y^2 are the same, that is, $\sigma_x^2 = \sigma_y^2 = \sigma^2$, the *variance ratio statistic F,* defined by

$$F = S_x^2 / S_y^2, \qquad ...(22.38)$$

where $\quad S_x^2 = \dfrac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2$ and $S_y^2 = \dfrac{1}{n_2 - 1} \sum_{i=1}^{n_2} (y_i - \bar{y})^2$

follows sampling distribution with probability density function given by

$$\frac{(\nu_1/\nu_2)^{\nu_1/2}}{B\left(\dfrac{\nu_1}{2}, \dfrac{\nu_2}{2}\right)} \cdot \frac{F^{(\nu_1/2)-1}}{\left(1 + \dfrac{\nu_1}{\nu_2} F\right)^{(\nu_1+\nu_2)/2}}, \quad 0 < F < \infty, \qquad ...(22.39)$$

where $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$.

The distribution defined by (22.39) is called *Snedecor's F-distribution with (ν_1, ν_2) degrees of freedom* and the variate F is denoted by $F_{(\nu_1, \nu_2)}$. Generally, the greater of the two variances S_x^2 and S_y^2 is taken as numerator and ν_1 corresponds to the variance in the numerator. The probability curve for the F-distribution is shown in Fig. 22.8.

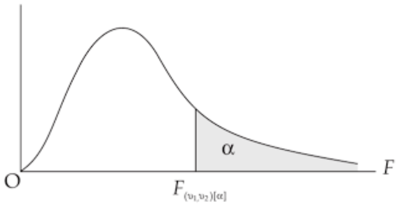


Fig. 22.8

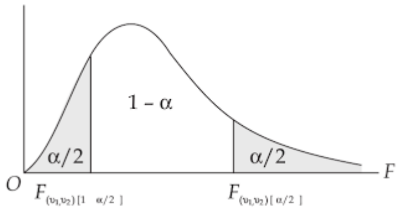


Fig. 22.9

The curve is not symmetric and the shape depends on the degrees of freedom ν_1 and ν_2 and their order. The curve is completely described by ν_1 and ν_2.

Critical values and test of significance. Let $F_{(\nu_1, \nu_2)[\alpha]}$ denote the value of F for (ν_1, ν_2) degrees of freedom such that the area to the right of this point is α, that is, $P[F > F_{(\nu_1, \nu_2)[\alpha]}] = \alpha$, as shown in Fig. 22.9. The Tables IV A & B (p. 530-31), give critical values or significant values of $F_{(\nu_1, \nu_2)[\alpha]}$ for the right-tailed test for different *df.* (ν_1, ν_2) and significant level $\alpha = 0.05$ and .01, respectively.

For an F-variate the reciprocal relation

$$F_{(\nu_1, \nu_2)[\alpha]} = \frac{1}{F_{(\nu_2, \nu_1)[1-\alpha]}}$$

also holds.

At a specific level α and degrees of freedom (ν_1, ν_2) the null hypothesis H_0: $\sigma_x^2 = \sigma_y^2$ is rejected against the alternate hypothesis,

(i) $H_1 : \sigma_x^2 > \sigma_y^2$ and $F = \dfrac{S_x^2}{S_y^2}$, if $F > F_{(\nu_1, \nu_2)[\alpha]}$

(ii) $H_1 : \sigma_x^2 < \sigma_y^2$ and $F = \dfrac{S_y^2}{S_x^2}$, if $F > F_{(\nu_2, \nu_1)[\alpha]}$

(iii) $H_1 : \sigma_x^2 \neq \sigma_y^2$ and $F = \dfrac{S_x^2}{S_y^2}$, if $F > F_{(\nu_1, \nu_2)[\alpha/2]}$

We must note that F-distribution is not symmetric but as in chi-square test, here also equal tails are used in two-tailed test as a matter of mathematical convenience only.

Here, we have

$n_1 = 10,\ n_2 = 12,\ \bar{x} = 15,\ \bar{y} = 14,\ \Sigma(x_i - \bar{x})^2 = 90,\ \Sigma(y_i - \bar{y})^2 = 108,$

$$S_1^2 = \frac{90}{9} = 10,\quad S_2^2 = \frac{108}{11} = 9.82, \text{ and } S^2 = \frac{1}{n_1+n_2-2}[\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2] = \frac{90+108}{20} = 9.9.$$

We test $H_0: \sigma_1^2 = \sigma_2^2$ against the right-tailed alternative $H_1: \sigma_1^2 > \sigma_2^2$. Under H_0, the statistic F given by

$$F = \frac{S_1^2}{S_2^2} = \frac{10}{9.82} = 1.018 \sim F_{(9.11)}.$$

From Table IV A, $F_{(9,\ 11)\ [.05]} = 2.90$. Since F calculated is less than the F tabulated, hence H_0 is accepted.

Since $H_0: \sigma_1^2 = \sigma_2^2$ is established, we can now apply t test for testing $H_0: \mu_1 = \mu_2$ against the alternative $H_1: \mu_1 \neq \mu_2$.

Under H_0, the statistic t given by

$$t = \frac{\bar{x} - \bar{y}}{S\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{15 - 14}{3.15\sqrt{\frac{1}{10} + \frac{1}{12}}} = \frac{1}{1.349} = 0.74 \sim t_{20}.$$

From Table II, $t_{20[.05]} = 2.086$. Since t calculated is less than the t-tabulated hence the hypothesis $H_0: \mu_1 = \mu_2$ may be accepted.

Since both the null hypotheses $H_0: \sigma_1^2 = \sigma_2^2$ and $H_0: \mu_1 = \mu_2$ are accepted so samples may be considered to come from the same normal population.

EXERCISE 22.4

1. A manufacturer claims that his measuring instrument has a variability measured by S.D. $\sigma = 2$. During a test the measurements recorded are 4.1, 5.2 and 10.2. Do these data confirm or disprove his claim? Construct a 90% confidence interval to estimate the true population variance.
2. A precision instrument is guaranteed to read accurately to within 2 units. A sample of four instrument readings on the same object yield the measurements 353, 351, 351 and 355. Test the null hypothesis that $\sigma = 0.7$ against the alternative $\sigma > 0.7$ at $\alpha = .05$.
3. Playing 10 rounds of golf on his home course, a golf professional averaged 71.3 with a S.D. 1.32. Test the null hypothesis at $\alpha = .05$ that consistency of his game on his home course is actually measured by $\sigma = 1.20$ against the alternative that he is less consistent.
4. Following data gives the amounts of sulphur monoxide recorded by two instruments A and B in the atmosphere. Assuming the populations of measurements to be normal, test the hypothesis $H_0: \sigma_A = \sigma_B$ against $H_1: \sigma_A \neq \sigma_B$ at $\alpha = .02$,

Instruments	*Amounts of sulphur monoxide*								
A	0.86	0.82	0.75	0.61	0.89	0.64	0.81	0.68	0.65
B	0.87	0.74	0.63	0.55	0.76	0.70	0.69	0.57	0.53.

5. The following are the values in thousands of an inch obtained by two technicians in 10 successive measurements with the same micrometer. Is one technician significantly more consistent than the other at $\alpha = 0.05$?

Technician A :	503	505	497	505	495	502	499	493	510	501
Technician B :	502	497	492	498	499	495	497	496	498.	

6. The nicotine contents in milligrams of two samples of tobacco were found to be as follows:

Sample A :	24	27	26	21	25	
Sample B :	27	30	28	31	22	36

Can it be claimed that two samples come from the same normal population?

7. For the two samples

105 108 86 103 103 107 124 105 and 89 92 84 97 103 107 111 97

giving the relative output of tin plate workers under two different working conditions, test the hypothesis at $\alpha = .05$, H_0: $\sigma_1^2 = \sigma_2^2$ against the alternative H_1: $\sigma_1^2 > \sigma_2^2$ assuming the two populations to be normal.

22.10 CHI-SQUARE TEST OF GOODNESS-OF-FIT, CONTINGENCY TABLES AND YATE'S CORRECTION FOR CONTINUITY

In many random experiments we are interested to know whether a particular probabilistic model is appropriate or not. For example, we may hypothesize that number of industrial accidents occurring monthly at a particular industrial plant follow Poission distribution. This hypothesis can be tested by observing the number of accidents over a sequence of months; finding the theoretical or expected number of accidents on the basis of the hypothesis made and then testing whether the deviations between the observed and the expected number of accidents in each category can be attributed to fluctuations of sampling or not.

The statistical tests that determine whether a given theoretical probability distribution is appropriate in case of the random phenomena under study are called 'goodness of fit' tests.

Suppose that $O_1, O_2 \ldots, O_k$ are the observed frequencies and $E_1, E_2, \ldots, E_k$ are the corresponding expected frequencies in the k categories on the basis of the hypothesis made. If hypothesis is correct, the observed cell frequency O_i should not be much different from the expected frequency E_i. The larger the difference, the more likely it is that the hypothesis is incorrect. The chi-square statistic to test the *goodness-of-fit* between the observed and the expected frequencies is defined as

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}, \qquad \text{...(22.40)}$$

where $\sum_{i=1}^{k} O_i = \sum_{i=1}^{k} E_i = n$, the total cell frequency.

When n is large the statistic χ^2 defined by (22.40) follows chi-square probability distribution with $\nu = k - m$ degrees of freedom, where k is the number of categories and m is the number of constraints applied to the observed data to calculate the expected frequencies.

If the theoretical cell frequencies are correct then χ^2 is close to zero but if theoretical cell frequencies are incorrect then χ^2 is large and thus we use right-tailed statistical test to find the significant value of χ^2 for the specified degrees of freedom ν and level of significance α.

Also to apply χ^2-test of goodness-of-fit, we pool some of the data so that no expected frequency is less than 5 and we change the degrees of freedom accordingly. This is done to avoid irregularities due to discontinuity since distribution of χ^2 is continuous but distribution of frequencies by nature is discontinuous.

Example 22.35: Suppose that a dice is tossed 120 times and the recorded data is as follow:

Face :	1	2	3	4	5	6
Observed frequency :	20	22	17	18	19	24

Test the hypothesis that the dice is unbiased at $\alpha = 0.05$.

Solution: On the basis of the null hypothesis that dice is unbiased the probability p_i for the face i is 1/6. So we test the hypothesis

$$H_0 : p_1 = p_2 = \ldots = p_6 = 1/6.$$

Thus, expected frequencies E_i for the face i is $np_i = 120 \times 1/6 = 20$, $i = 1, 2, \ldots, 6$

Under the hypothesis H_0, the statistic $\chi^2 = \sum_{i=1}^{6} \frac{(O_i - E_i)^2}{E_i}$, follows χ^2 distribution with degrees of freedom $\nu = k - m = 6 - 1 = 5$.

We have, $$\chi^2 = \frac{0+4+9+4+1+16}{20} = \frac{34}{20} = 1.7.$$

From Table III, $\chi^2_{\nu[.05]} = 11.07$. Since χ^2 calculated is less than the χ^2 tabulated the hypothesis may be accepted , that is, dice may be considered to be unbiased.

Example 22.36: The proportion of blood phenotypes A, B, AB and O in a population are expected to be 0.41, 0.10, 0.04 and 0.45, respectively. To determine whether or not the actual proportions fit this set of probabilities, a random sample of size 200 is selected from this population and blood phenotypes of the units selected are recorded. The observed data is given as follows:

Phenotypes :	*A*	*B*	*AB*	*O*
No. of units :	89	18	12	81

Test the goodness-of-fit of these blood phenotype proportions at $\alpha = .05$.

Solution: The hypothesis to be tested is

$H_0 : p_A = 0.41, \quad p_B = 0.10, \quad p_{AB} = 0.04, \quad p_O = 0.45.$

Under H_0, the expected cell frequencies are

$$E(A) = 200(0.41) = 82, \quad E(B) = 200\,(0.10) = 20$$
$$E(AB) = 200\,(0.04) = 8, \quad E(O) = 200\,(0.45) = 90.$$

Thus $\chi^2 = \sum_{i=1}^{4} \frac{(O_i - E_i)^2}{E_i} = \frac{49}{82} + \frac{4}{20} + \frac{16}{8} + \frac{81}{90} = 3.70$

Number of degrees of freedom = 4 – 1 = 3.

From Table III, $\chi^2_{3[.05]} = 7.82$. Since χ^2 calculated is less than χ^2 tabulated, thus null hypothesis H_0 may be accepted at 5% level of significance.

Example 22.37: During 400 five-minute interval the air traffic control of an airport received 0, 1, 2, …, or 13 radio messages with respective frequencies of 3, 15, 47, 76, 68, 74, 46, 39, 15, 9, 5, 2, 0 and 1. Test at $\alpha = 0.05$, the hypothesis that the number of radio messages received during a 5 minute interval follow Poisson distribution with $\lambda = 4.6$.

Solution: Let the r.v. X be the number of radio messages received during a 5-minute interval and p_x be the probability of receiving x messages. Set the null hypothesis that X follows a Poisson distribution with parameter 4.6. Thus, we test the hypothesis.

$$H_0 : p_x = e^{-4.6} \frac{(4.6)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Form the following table:

No. of radio messages (x)	*Observed frequencies (O)*	*Poisson probabilities (p_x)*	*Expected frequencies (E = 400 p_x)*
0	3	0.010	4.0
1	15	0.046	18.4
2	47	0.107	42.8
3	76	0.163	65.2
4	68	0.187	74.8
5	74	0.173	69.2
6	46	0.132	52.8
7	39	0.087	34.8
8	15	0.050	20.0
9	9	0.025	10.0
10	5	0.012	4.8
11	2	0.005	2.0
12	0	0.002	0.8
13	1	0.001	0.4

To apply χ^2-test of goodness-of-fit, since no expected frequency should be less than 5, so we pool the first two expected frequencies and the last four expected frequencies. The modified frequencies are:

Observed (O)	:	18	47	76	68	74	46	39	15	9	8
Expected (E)	:	22.4	42.8	65.2	74.8	69.2	52.8	34.8	20.0	10.0	8.0.

Thus $\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$

$$= \frac{(4.4)^2}{22.4} + \frac{(4.2)^2}{42.8} + \frac{(10.8)^2}{65.2} + \frac{(6.8)^2}{74.8} + \frac{(4.8)^2}{69.2} + \frac{(5.8)^2}{52.8} + \frac{(4.2)^2}{34.8}\ \frac{(5)^2}{20} + \frac{(1)^2}{10} + \frac{(0)^2}{8}$$

$= 6.749$

Number of degrees of freedom = 10 – 1 = 9.

From Table III, $\chi^2_{9[.05]} = 16.919$. Since χ^2 calculated is less than the χ^2 tabulated so null hypothesis may be accepted at 5% level of significance.

22.10.1 Chi-Square Test of Independence in Contingency Tables

Sometimes experimental units are classified according to two characteristics generating a bivariate data. The resulting observations are displayed in the form of a two-way table, called a *contingency table*, consisting of finite numbers of rows and columns. One characteristic varies along the rows and the second characteristic varies along the columns.

For example, a random sample of 500 employees of a PSU are classified whether they are in a low, medium, or high income bracket and whether or not they favour the new salary structure announced. The data can be presented in the form of the following 2×3 contingency table.

Salary structure	*Income level*			*Total*
	Low	*Medium*	*High*	
For	91	106	98	295
Against	80	72	53	205
Total	171	178	151	500

A contingency table with r rows and c columns is referred to as an $r \times c$ table. The row and column totals are called the *marginal frequencies*.

In two categorical variable data, our interest may be to know whether or not the two characteristics are independent. The chi-square test procedure can be used to test the hypothesis for independence of two characteristics of classification. We test the null hypothesis

H_0: The two characteristics of classification are independent, against the alternative

H_1: The two characteristics of classification are dependent.

Let O_{ij} be the observed cell frequency in row i and column j of the contingency table and if we know E_{ij} the expected cell frequency under H_0, then we can use χ^2 to compare the observed and expected frequencies.

Let p_{ij} be the probability of falling observation in the ith row and jth column and if n is the total number of observations, then

$$E_{ij} = np_{ij} = n\, p_i\, q_j$$

since under hypothesis of independence, $p_{ij} = p_i q_j$, where p_i and q_j are the marginal probabilities of falling observations in the ith row and ith column respectively.

We approximate p_i with $\hat{p}_i = \frac{n_i}{n}$ and q_j with $\hat{q}_j = \frac{m_j}{n}$, where n_i, m_j respectively are the ith row and jth column totals. Thus estimated expected cell frequencies under H_0 become

$$E_{ij} = n\frac{n_i}{n}\frac{m_j}{n} = \frac{n_i m_j}{n},$$

and the statistic χ^2 is given by

$$\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}}. \qquad ...(22.41)$$

This test statistic χ^2 can be shown to have an approximate χ^2 probability distribution with $(r - 1)(c - 1)$ degrees of freedom and the hypothesis H_0 is tested accordingly using the right-tailed test.

Example 22.38: A company operates four machines on three separate shifts daily. The following table presents the data for machine breakdowns resulted during a 6-month time period.

Shift	*Machine*				*Total*
	A	*B*	*C*	*D*	
1	10	12	6	7	35
2	10	24	9	10	53
3	13	20	7	10	50
Total	33	56	22	27	138

Test the hypothesis that for an arbitrary breakdown the machine causing the breakdown and the shift on which the breakdown occurred are independent.

Solution: Let H_0: For an arbitrary breakdown the machine and the shift are independent.

Under H_0, the twelve expected frequencies are given by

$E_{11} = \frac{35 \times 33}{138} = 8.37,\quad E_{12} = \frac{35 \times 56}{138} = 14.20,\quad E_{13} = \frac{35 \times 22}{138} = 5.58,$

$E_{14} = \frac{35 \times 27}{138} = 6.85\quad E_{21} = \frac{53 \times 33}{138} = 12.67,\quad E_{22} = \frac{53 \times 56}{138} = 21.50,$

$E_{23} = \frac{53 \times 22}{138} = 8.45,\quad E_{24} = \frac{53 \times 27}{138} = 10.37,\quad E_{31} = \frac{50 \times 33}{138} = 11.96,$

$E_{32} = \frac{50 \times 56}{138} = 20.29,\quad E_{33} = \frac{50 \times 22}{138} = 7.79,\quad E_{34} = \frac{50 \times 27}{138} = 9.78.$

Under H_0, the statistic χ^2 is given by

$$\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2_{(r-1)(c-1)}$$

Thus, $$\chi^2 = \frac{(10-8.37)^2}{8.37} + \frac{(12-14.20)^2}{14.20} + \frac{(6-5.58)^2}{5.58} + \frac{(7-6.85)^2}{6.85}$$
$$+ \frac{(10-12.67)^2}{12.67} + \frac{(24-21.50)^2}{21.50} + \frac{(9-8.45)^2}{8.45} + \frac{(10-10.37)^2}{10.37}$$
$$+ \frac{(13-11.96)^2}{11.96} + \frac{(20-20.29)^2}{20.29} + \frac{(7-7.97)^2}{7.97} + \frac{(10-9.78)^2}{9.78} = 1.78$$

The *df*. are (2 – 1) (4 – 1) = 3.

From Table III, $\chi^2_{3\,[.05]} = 7.8$. Since χ^2 calculated is less than χ^2 tabulated for 3 *df* at $\alpha = .05$, hence null hypothesis may be accepted.

Example 22.39: To know about the student's response over the proposed evaluation system it was decided to select 200 1st year, 150 2nd year and 150 3rd year students from a college and record whether they are for, against or undecided for the proposed evaluation system. The observed responses were:

Evaluation System	*Students*			*Total*
	1st year	*2nd year*	*3rd year*	
For	82	70	62	214
Against	93	62	67	222
Undecided	25	18	21	64
Total	200	150	150	500

Test the hypothesis that the three categories of the students are homogeneous with respect to their opinions on the proposed evaluation system.

Solution: Let H_0: Students are homogeneous with respect to their opinions on the proposed evaluation system.

Assuming homogeneity, the expected cell frequencies are:

$$E_{11} = \frac{200 \times 214}{500} = 85.6, \quad E_{12} = \frac{150 \times 214}{500} = 64.2, \quad E_{13} = \frac{150 \times 214}{500} = 64.2,$$

$$E_{21} = \frac{200 \times 222}{500} = 88.8, \quad E_{22} = \frac{150 \times 222}{500} = 66.6, \quad E_{23} = \frac{150 \times 222}{500} = 66.6,$$

$$E_{31} = \frac{200 \times 64}{500} = 25.6, \quad E_{32} = \frac{150 \times 64}{500} = 19.2, \quad E_{33} = \frac{150 \times 64}{500} = 19.2.$$

Thus, $\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$

$$= \frac{(3.6)^2}{85.6} + \frac{(5.8)^2}{64.2} + \frac{(2.2)^2}{64.2} + \frac{(4.2)^2}{88.8} + \frac{(4.6)^2}{66.6} + \frac{(0.4)^2}{66.6} + \frac{(0.6)^2}{25.6} + \frac{(1.2)^2}{19.2} + \frac{(1.8)^2}{19.2}$$

$= 1.53.$

Degrees of freedom $\nu = (r - 1)(c - 1) = (3 - 1)(3 - 1) = 4.$

From Table IV $\chi^2_{4[0.5]} = 9.488$. Since χ^2 calculated is less than the χ^2 tabulated, hypothesis may be accepted at 5% level of significance.

Example 22.40: Two different sampling techniques were adopted while investigating the same group of students to find the number of students falling in different intelligence level. The results are tabulated as follow:

Techniques	*No. of Students*				
	Below average	*Average*	*Above average*	*Genius*	*Total*
X	86	60	44	10	200
Y	40	33	25	2	100
Total	126	93	69	12	300

Are the sampling techniques adopted significantly different?

Solution: Let us assume that sampling techniques are not significantly different thus we test H_0: Data obtained is independent of the sampling techniques adopted.

Under H_0, the expected frequencies are

$$E_{11} = \frac{126 \times 200}{300} = 84, \quad E_{12} = \frac{93 \times 200}{300} = 62, \quad E_{13} = \frac{69 \times 200}{300} = 46, \quad E_{14} = \frac{12 \times 200}{300} = 8,$$

$$E_{21} = \frac{126 \times 100}{300} = 42, \quad E_{22} = \frac{93 \times 100}{300} = 31, \quad E_{23} = \frac{69 \times 100}{300} = 23, \quad E_{24} = \frac{12 \times 100}{300} = 4.$$

Since to apply χ^2 test no expected cell frequency should be less than 5 but here $E_{24} = 4$, so we pool E_{24} with E_{23} (or, with E_{14}) and accordingly O_{24} with O_{23} (or with O_{14}). We have

$$\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

$$= \frac{(86-84)^2}{84} + \frac{(60-62)^2}{62} + \frac{(44-46)^2}{46} + \frac{(10-8)^2}{8} + \frac{(40-42)^2}{42} + \frac{(33-31)^2}{31} + \frac{[(25+2)-(23+4)]^2}{23+4}$$

$= 0.92.$

Degrees of freedom $\nu = (2 - 1)(4 - 1) - 1 = 2.$

From Table III, $\chi^2_{2[.05]} = 5.991$. Since χ^2 calculated is less than χ^2 tabulated, thus the null hypothesis H_0 is accepted at 5% level. Hence there is no significant difference in the sampling techniques adopted.

22.10.2 Yate's Correction for Continuity

The 2×2 contingency tables are of great practical importance. However, in a 2×2 table there is only one degrees of freedom and the frequency of only one cell can be assigned arbitrarily, but in case any of the expected frequency is less than 5, then pooling that cell frequency results in χ^2 with zero degree of freedom which is meaningless. In this case we apply a correction due to *F*. Yates known as *Yate's correction for continuity*. It consists in adding 0.5 to the cell frequency which is less than 5 and then adjusting for the remaining cell frequencies accordingly. The χ^2 test is applied in the resultant table without making any further correction.

Example 22.41: Two batches each of 12 experimental animals 'inoculated' and the other, 'not inoculated', were exposed to the infection of a disease. The following frequencies of dead and surviving animals were noted in the two cases, can the inoculation be regarded as effective against the disease?

Animals	*Dead*	*Survived*	*Total*
Inoculated	2	10	12
Not inoculated	8	4	12
Total	10	14	24

Solution: Let H_0: Inoculation is not effective against the disease.

Since, the cell frequencies are less than 5, applying Yate's correction for continuity the corrected table is

Animals	*Dead*	*Survived*	*Total*
Inoculated	2.5	9.5	12
Not inoculated	7.5	4.5	12
Total	10	14	24

Under H_0 the expected frequencies are

$$E_{11} = \frac{10 \times 12}{24} = 5, \quad E_{12} = \frac{14 \times 12}{24} = 7, \quad E_{21} = \frac{10 \times 12}{24} = 5, \quad E_{22} = \frac{14 \times 12}{24} = 7.$$

$$\text{Hence, } \chi^2 = \frac{(2.5)^2}{5} + \frac{(2.5)^2}{7} + \frac{(2.5)^2}{5} + \frac{(2.5)^2}{7} = 4.286$$

Degrees of freedom, $\nu = (2 - 1)(2 - 1) = 1$.

From Table III, $\chi^2_{1[.05]} = 3.841$. Since χ^2 calculated is greater than χ^2 tabulated, thus null hypothesis is rejected at 5% level of significance, that is, inoculation may be considered to be effective against the disease.

Response	College				Total
	A	B	C	D	
For	65	66	40	34	205
Against	42	30	33	42	147
Undecided	93	54	27	24	198
Total	200	150	100	100	550

Test for homogeneity of responses among the four colleges concerning student's uniform in the professional colleges.

7. A random chosen group of 20,000 non-smokers and one of 10,000 smokers were observed over a 10-year period. The following data relate the numbers of them that developed lung cancer during that period.

	Smokers	Non-smokers	Total
Lung cancer	62	14	76
No lung cancer	9,938	19,986	29,924
Total	10,000	20,000	30,000

At $\alpha = 0.01$, test the hypothesis that smoking and lung cancer are independent.

8. To study whether or not the level of earning is affected by educational attainment, a social scientist randomly selected 100 people from each of three income categories 'low', 'middle', 'high' and then recorded their educational attainment as in the following table:

Educational attainment	Income categories			Total
	Low	Middle	High	
No college	32	20	23	75
UG	13	16	1	30
PG	43	51	60	154
Doctoral	12	13	16	41
Total	100	100	100	300

Do these data indicate that the level of earning is affected by educational attainment? Test at $\alpha = 0.01$.

9. In an experiment on immunization of cattle from tuberculosis the following results were obtained:

	Affected	*Unaffected*
Inoculated	12	26
Not inoculated	16	06

Examine the effect of vaccine in controlling susceptibility to tuberculosis.

10. In an experiment on the immunization of goats from anthrax the following results were obtained:

	Died	*Survived*
Inoculated	2	10
Not inoculated	6	6

Give your conclusion on the efficiency of the vaccine.

ANSWERS

Exercise 22.1 (p. 481)

1. Yes
3. No
4. (a) 0.023 (b) 0.0038
5. 0.0179 (b) 0.740 (c) 0.242.
6. (a) Approximately normal with mean 0.75 and S. E. 0.0306
   (b) 0.0516 (c) .69 to .81.

Exercise 22.2 (p. 497)

1. $z = -1.25$, rejected
2. $z = -1.06$, rejected
3. (8.61, 15.38)
4. 20% to 29%
5. $z = 0.37$, accepted
6. $z = 2.5$, unlikely to be hidden
7. $z = 2.38$, second production line does superior work
8. $z = 1.489$, accepted
9. 0.0013
10. $z = -2.681$, claim may be considered to be valid.
11. $n = 35$
12. (–1022, 4622) interval contains zero, cannot conclude that type B is superior to type A
13. (2.80, 3.40), yes
14. $z = 3.535$, yes
15. (a) .9772 (b) 0.0062.

Exercise 22.3 (p. 508)

1. $t = 1.32$, not viable
2. 41.5 ± 1.6
3. (a) 7.496 (b) $t = -2.849$, reject H_0 (c) yes
4. $t = 2.89$, yes!
5. $t = -0.609$; don't differ significantly
6. $t = 4.03$; program is effective, [4.0, 6.4]
7. $t = 0.95$; accepted
8. (i) $t = 2.17$; B is superior (ii) $t = 5.09$; significant (iii) 0.2 ± 0.075
10. 0.47.

Exercise 22.4 (p. 514)

1. $\chi^2 = 5.29$, claim accepted, (3.53, 206.07)
2. $\chi^2 = 22.45$, reject H_0
3. $\chi^2 = 10.89$, H_0 accepted
4. $F = 1.15$, H_0 accepted
5. $F = 2.4$, No
6. $H_0 : \mu_1 = \mu_2$; $t = 1.9$ not significant
   $H_0 : \sigma_1^2 = \sigma_1^2$; $F = 4.08$ not significant, yes!
7. $F = 1.26$ accepted.

Exercise 22.5 (p. 523)

1. $\chi^2 = 58.542$, H_0 rejected
2. $\chi^2 = 19.63$, H_0 rejected
3. $\chi^2 = 3.05$, H_0 accepted
4. $\chi^2 = 40.937$, H_0 rejected
5. $\chi^2 = 20.179$, H_0 rejected
6. $\chi^2 = 31.17$, Non-homogeneous response
7. $\chi^2 = 79.83$, H_0 rejected
8. $\chi^2 = 19.172$, yes!
9. Vaccine is effective
10. Vaccine is efficient.

Appendix II

Additional Proofs. Section 20.9 (p. 322) Laplace Equation in Polar Coordinates

We have

$$x = r\cos\theta,\ y = r\sin\theta. \text{ These give } r = \sqrt{(x^2+y^2)},\ \theta = \tan^{-1}(y/x)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2+y^2)}} = \cos\theta \quad \text{and} \quad \frac{\partial\theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

Thus, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial\theta}\cdot\dfrac{\partial\theta}{\partial x} = \cos\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial\theta}$, which implies

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}. \text{ Similarly, } \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Also, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \left(\cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta}\right)\left(\cos\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial\theta}\right)$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}, \quad \text{...(i)}$$

and $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \left(\sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta}\right)\left(\sin\theta\dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial\theta}\right)$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} \quad \text{...(iii)}$$

Adding (i) and (ii) we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Thus the Laplace equation in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial\theta^2} = 0.$$

Section 20.14 (p.355) Associated Legendre Functions

The equation (20.276) is

$$\Phi'' + (\cot\phi)\,\Phi' + [n(n+1) - m^2 \operatorname{cosec}^2\phi]\Phi = 0 \qquad \text{...(i)}$$

Set $\alpha = \cos\phi$, we have

$$\frac{d\Phi}{d\phi} = \frac{d\Phi}{d\alpha}\frac{d\alpha}{d\phi} = -\sin\phi\frac{d\Phi}{d\alpha}$$

$$\frac{d^2\Phi}{d\phi^2} = \frac{d}{d\phi}\left(-\sin\phi\frac{d\Phi}{d\alpha}\right) = -\cos\phi\frac{d\Phi}{d\alpha} + \sin^2\phi\frac{d^2\Phi}{d\alpha^2}$$

$$= -\alpha\frac{d\Phi}{d\alpha} + (1-\alpha^2)\frac{d^2\Phi}{d\alpha^2}$$

Substituting these in (i) and simplifying we obtain

$$(1-\alpha^2)\frac{d^2\Phi}{d\alpha^2} - 2\alpha\frac{d\Phi}{d\alpha} + \left[n(n+1) - \frac{m^2}{1-\alpha^2}\right]\Phi = 0 \qquad \text{...(ii)}$$

Again substituting $\Phi(\alpha) = (1-\alpha^2)^{m/2}v(\alpha)$, (ii) reduces to

$$(1-\alpha^2)v'' - 2(m+1)\alpha v' + [n(n+1) - m(m+1)]v = 0. \qquad \text{...(iii)}$$

Legendre's equation of parameter n is

$$(1-\alpha^2)y'' - 2\alpha y' + n(n+1)y = 0 \qquad \text{...(iv)}$$

Differentiating it m times w.r.t. α and simplifying we obtain

$$(1-\alpha^2)\,y^{(m+2)} - 2\alpha y^{(m+1)} + [n(n+1) - m(m+1)]y^{(m)} = 0 \qquad \text{...(v)}$$

This is same as equation (iii) for $v = y^{(m)} = \dfrac{dy^m}{d\alpha^m}$ and hence solution of (iii) is

$$v = \frac{d^m P_n}{d\alpha^m},$$

where P_n is Legendre function.

The corresponding function

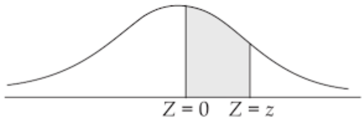
$$\Phi(\alpha) = (1-\alpha^2)^{m/2}v(\alpha) = (1-\alpha^2)^{m/2}\frac{d^m P_n}{d\alpha^m} = P_n^m(\alpha)$$

is called an *associated Legendre function*.

Appendix I

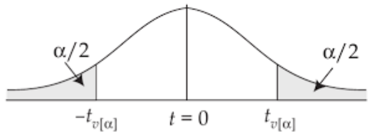
Table I: Areas Under The Standerd Normal Curve

$$P(0 < Z < z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}z^2} dz$$



↓z→	0	1	2	3	4	5	6	7	8	9
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0759
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3655	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4637
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4678	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4959	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

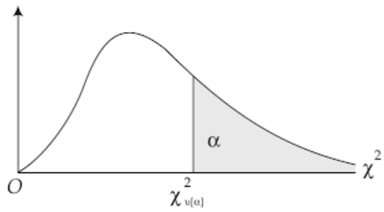
Table II: Critical Values $t_{\nu[\alpha]}$ of t-Distribution (Two-Tail Areas) $P[\,|t| > t_{\nu[\alpha]}] = \alpha$



d.f. (ν)	*Level of significance* (α)					
	0.50	0.10	0.05	0.02	0.01	0.001
1	1.00	6.31	12.71	31.82	63.86	636.62
2	0.82	2.92	4.30	6.97	6.93	31.60
3	0.77	2.35	3.18	4.54	5.84	12.94
4	0.74	2.13	2.78	3.75	4.60	8.61
5	0.73	2.02	2.57	3.37	4.03	6.86
6	0.72	1.94	2.45	3.14	3.71	5.96
7	0.71	1.90	2.37	3.00	3.50	5.41
8	0.71	1.86	2.31	2.90	3.36	5.04
9	0.70	1.83	2.26	2.82	3.25	4.78
10	0.70	1.81	2.23	2.76	3.17	4.59
11	0.70	1.80	2.20	2.72	3.11	4.44
12	0.70	1.78	2.18	2.68	3.06	4.32
13	0.69	1.77	2.16	2.65	3.01	4.22
14	0.69	1.76	2.15	2.62	2.98	4.14
15	0.69	1.75	2.13	2.60	2.95	4.07
16	0.69	1.75	2.12	2.58	2.92	4.02
17	0.69	1.74	2.11	2.57	2.90	3.97
18	0.69	1.73	2.10	2.55	2.88	3.92
19	0.69	1.73	2.09	2.54	2.86	3.88
20	0.69	1.73	2.09	2.53	2.85	3.85
21	0.69	1.72	2.08	2.52	2.83	3.83
22	0.69	1.72	2.07	2.51	2.82	3.79
23	0.69	1.71	2.07	2.50	2.81	3.77
24	0.69	1.71	2.06	2.49	2.80	3.75
25	0.68	1.71	2.06	2.49	2.79	3.73
26	0.68	1.71	2.06	2.48	2.78	3.71
27	0.68	1.70	2.05	2.47	2.77	3.69
28	0.68	1.70	2.05	2.47	2.76	3.67
29	0.68	1.70	2.05	2.46	2.76	3.66
30	0.68	1.70	2.04	2.46	2.75	3.65
∞	0.67	1.65	1.96	2.33	2.58	3.29

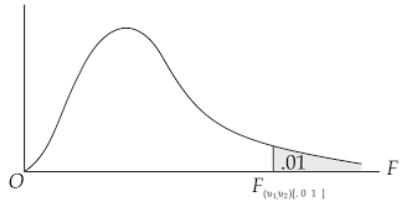
Table III: Critical Values $\chi^2_{\upsilon[\alpha]}$ Of Chi-Square Distibution (Right Tail Areas)

$P[\chi^2_\nu > \chi^2_{\nu[\alpha]}] = \alpha$



Degree of freedom (ν)	Level of significance (α) 0.995	0.99	0.975	0.95	0.05	0.025	0.01	0.005
1	0.000	0.000	0.001	0.004	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	11.070	12.832	15.086	16.750
6	0.676	0.872	1.237	1.634	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	18.360	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	22.362	24.736	24.888	29.819
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796
23	9.260	10.196	11.688	13.091	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672

Table IV B: CRITICAL VALUES OF THE F-DISTRIBUTION (RIGHT TAIL AREAS) $F_{(\nu_1, \nu_2)[.01]}$



ν_2 \ ν_1	1	2	3	4	5	6	8	12	24	∞
1	4052	4999.5	5403	5625	5764	5859	5982	6106	6235	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.37	99.42	99.46	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.29	9.89	9.47	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	12.25	9.95	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.30	3.96	3.59	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.14	3.80	3.43	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.79	3.46	3.08	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.51	3.17	2.80	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.23	2.90	2.52	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.20	2.87	2.49	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01

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